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Geometrical theory of fluid flows and dynamical systems

Tsutomu Kambe *

Institute of Dynamical Systems, Higashi-yama 2-11-3, Meguro-ku, Tokyo 153-0043, Japan

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Abstract

Various dynamical systems have often common geometrical structures and can be formulated on the basis of Riemannian geometry and Lie group theory, provided that a dynamical system has a group symmetry, namely it is invariant under group transformations, and further that the group manifold is endowed with a Riemannian metric. The basic ideas and tools are described, and their applications are presented for the following five problems: (a) free rotation of a rigid body, which is a well-known system in mechanics and presented as an illustrative example of the geometrical theory; (b) geodesic equation and KdV equation on the group of diffeomorphisms of a circle and its extended group; (c) a self-gravitating system of a finite number of points masses and a geometrical interpretation of chaos of Hénon–Heiles system; (d) geometrical formulation of hydrodynamics of an incompressible ideal fluid on the group of volume preserving diffeomorphisms, where an interpretation of the origin of Riemannian curvatures of the fluid flow is given; (e) geodesic equation on a loop group and the local induction equation for the motion of a vortex filament, where the geodesic equation on its extended group is found to be equivalent to the equation for a vortex filament with an axial flow along it.

It is remarkable that the present geometrical formulations are successful for all the problems considered here and give an insight into the deep background common to the diverse physical systems. Furthermore, the geometrical formulation opens a new approach to various dynamical systems, which is rewarded with new results. © 2002 Elsevier Science B.V. All rights reserved.

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* Fax.: +81-3-3792-2859.
E-mail address: kambe@gate01.com (T. Kambe).

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1. Introduction

Dynamical systems can be formulated in general on the basis of the Riemannian geometry and Lie algebra. In this review, the mathematical features are illustrated and physical aspects are exemplified by five systems. (a) Free rotation of a rigid body (Euler's top), which is presented as an illustrative example. This is well-known in physics and one of the simplest systems of finite degrees of freedom. (b) Derivations of a geodesic equation on a group of diffeomorphisms of a circle and KdV equation on its extended group, in which one of the soliton equations is derived on a geometric framework. (c) Geometrical analysis of chaos of a Hamiltonian system, which is a self-gravitating system of a finite number of point masses. (d) Geometrical formulation of hydrodynamics of an incompressible ideal fluid, which gives not only geometrical characterization of flows but also interpretation of the origin of Riemannian curvatures of the fluid flow. (e) Derivation of a geodesic equation on a loop group leads to the local induction equation of the motion of a vortex filament, and the equation on its extended group is found to be equivalent to the equation for a vortex filament with an axial flow along it.

In Section 2, an introductory review is given for flows and diffeomorphisms, vectors and forms, Lie group and Lie algebra. In Section 3, the theory of Riemannian differential geometry is described compactly and basic concepts are presented: the first and second fundamental forms, affine connection, geodesic equation, Jacobi field, and Riemannian curvatures. Full accounts are found in the textbooks by Frankel (1997), Flanders (1963), Arnold (1978), and Abraham and Marsden (1978), although some parts are original (e.g. Section 3.7).

In Sections 4–8, typical five dynamical systems are reformulated according to the mathematical framework presented in the preceding sections. The governing equation of each system is now obtained as a geodesic equation on a group manifold associated with the individual system. It is to be noted that, although the governing equations are derived already in the physics, the present derivations are new and based on very general setup and concepts of metric, connection and Lie algebra in the Riemannian differential geometry. Sections 4–6 are reviews of published works: Suzuki et al. (1998) (Section 4); Ovsienko and Khesin (1987), Misiołek (1997) and Kambe (1998) (Section 5); and Cerruti-Sola and Pettini(1996) (Section 6), respectively. An original Appendix A.3 is added to supplement the descriptions of Section 5 about the two-cocycle, central extension and Bott cocycle.

One of the aims of this note is to give a geometrical framework to the description of flows of an ideal incompressible fluid and an interpretation to the origin of Riemannian curvatures of the flows in Section 7, based on Misiołek (1993), Nakamura et al. (1992), and Hattori and Kambe (1994). Further, Section 8 describes a new formulation for the geodesic equations of motion of a vortex filament on the basis of the theory of loop group and its extension. The associated loop algebra leads to the Landau–Lifshitz equation, which is derived as the geodesic equation (Section 8.2) and equivalent to the local induction equation. Further the loop-group formulation admits a central extension, known as the Kac–Moody algebra in the gauge theory of theoretical physics (de Azcárraga and Izquierdo, 1995) and leads to the equation of Fukumoto and Miyazaki (1991). This gives a new interpretation to the equations of a vortex filament from a geometrical point of view.

The present review article is based on the lecture note prepared during the author's stay at the Isaac Newton Institute in the year 2000 (September–December) which resulted in the preprint (Kambe, 2001). Before describing the details of particular dynamical systems, mathematical concepts are presented first and reviewed briefly.

2. Flows, diffeomorphisms and Lie groups

2.1. Differentiable map and diffeomorphism

A *manifold* M^n is an n-dimensional space that is locally $\mathbb{R}^n$ in the sense described just below, but is not necessarily a Euclidean space $\mathbb{R}^n$ itself. For example, consider a sphere S^2 which is a two-dimensional object embedded in three-dimensional Euclidean space $\mathbb{R}^3$. The two-sphere S^2 is not considered to be a part of the Euclidean plane $\mathbb{R}^2$. However, an observer on S^2 would see that the immediate neighborhood is described by two coordinates and cannot be distinguished from a small domain of $\mathbb{R}^2$.

In general, an n-dimensional manifold M^n is a topological space,[1] which is covered with a collection of open subsets U_1, $U_2, \ldots$ such that each point of M^n lies in at least one of them. Using a map F_U called a *homeomorphism*,[2] each open subset U is one-to-one correspondence with an open subset $F_U(U)$ of $\mathbb{R}^n$. Each pair (U, F_U) defines a *coordinate patch* on M. To each point p $(\in U \subset M)$, we may assign the n coordinates of the point $F_U(p)$ in $\mathbb{R}^n$. For this reason, we call F_U a *coordinate map* with the jth component written as x^j_U. This is often described in the following way. On the patch U, a point p is represented by a *local coordinates*, $p = (x^1_p, \ldots, x^n_p)$. Suppose that the patch U with the local coordinates $x = (x^1, \ldots, x^n)$ overlap with another patch V with the local coordinates $y = (y^1, \ldots, y^n)$. Then, a point p lying in the overlapping domain can be represented by both systems of x and y. In particular, y^i is expressed in terms of x as

$$y^i = y^i(x^1, \ldots, x^n), \quad (i = 1, \ldots, n). \tag{1}$$

We require that these functions are smooth and differentiable, and that the Jacobian determinant

$$|J| = \frac{\partial(y)}{\partial(x)} = \frac{\partial(y^1, \ldots, y^n)}{\partial(x^1, \ldots, x^n)} \tag{2}$$

does not vanish at any point $p \in U \cap V$ (Flanders, 1963, Chapter V).

Let $F : M^n \to W^r$ be a smooth map from a manifold M^n to another W^r. In local coordinates $x = (x^1, \ldots, x^n)$ in the neighborhood of the point $p \in M^n$ and $z = (z^1, \ldots, z^r)$ in the neighborhood of $F(p)$ on W^r, the map F is described by r functions $F^i(x)$, $(i = 1, \ldots, r)$ of n variables, abbreviated to $z = F(x)$ or $z = z(x)$, where F^i are differentiable functions of x^j $(j = 1, \ldots, n)$.

[1] A *topological space* is a set M with a collection of subsets called *open* sets. An example of open sets is a *ball* in the Euclidean space $\mathbb{R}^n$ defined by $B_a(\varepsilon) = \{x \in \mathbb{R}^n \mid \|x - a\| < \varepsilon,\ a \in \mathbb{R}^n\}$. In its generalization, the open sets satisfy the followings: (i) If U and V are open, so is their intersection $U \cap V$. (ii) The union of any collection of open sets (possibly infinite in number) is open. (iii) The empty set is open. (iv) The set M is open, a generalization of the entire $\mathbb{R}^n$ which is open. The topology of $\mathbb{R}^n$ is said to be induced by the euclidean norm $\|\cdot\|$. The topology of M is defined by the open subsets. A subset of M is said *closed* if its complement is open.

[2] A *homeomorphism* F takes an open set M into an open set N in the following sense. Namely, $F : M \to N$ is one-to-one and onto (thus the inverse map $F^{-1} : N \to M$ exists). In addition, both F and F^{-1} are continuous.

When $n = r$, we say that the map F is a *diffeomorphism*, provided F is differentiable (thus continuous), one-to-one, onto, and in addition F^{-1} is differentiable. Such an F is a *differentiable homeomorphism*. If the inverse F^{-1} does exist and the Jacobian determinant does not vanish, then the inverse function theorem would assure us that the inverse is differentiable. In the next section, the fluid flow is described to be a smooth sequence of diffeomorphisms of particle configuration (of infinite dimension).

2.2. *Flows and vector fields*

2.2.1. *A flow and its velocity field*

Given a flow of a fluid in $\mathbb{R}^3$, one can construct a 1-parameter family of maps: $\phi_t : \mathbb{R}^3 \to \mathbb{R}^3$, where ϕ_t takes a fluid particle located at p when $t = 0$ to the position of the same fluid particle $\phi_t(p)$ at a later time $t > 0$. The family of maps are the so-called *Lagrangian* representation of motion of fluid particles. In terms of local coordinates, the jth coordinate of the particle is written as $x^j \circ \phi_t(p) = x_t^j(p)$, where '$x^j \circ$' means the coordinate map.

Associated with any such flow, we have a velocity at p,

$$v(p) := \left. \frac{\mathrm{d}}{\mathrm{d}t} \phi_t(p) \right|_{t=0}.$$

In terms of the coordinates, we have $v^j(p) = (\mathrm{d}x_t^j(p)/\mathrm{d}t)|_{t=0}$. Taking a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ and differentiating $f(\phi_t(p))$ with respect to t, we have [3]

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} f(\phi_t(p)) \right|_{t=0} = \sum_j \frac{\mathrm{d}x_t^j}{\mathrm{d}t} \frac{\partial f}{\partial x^j} = \sum_j v^j(p) \frac{\partial}{\partial x^j} f =: X(p) f,$$

$$X(p) := \sum_j v^j(p) \frac{\partial}{\partial x^j}. \tag{3}$$

This is also written in the following way by considering that f is an operator,

$$Xf = \frac{\mathrm{d}}{\mathrm{d}t} f(\phi_t) := \frac{\mathrm{d}}{\mathrm{d}t} f \circ \phi_t. \tag{4}$$

The differential operator X is written also as v by the reason described in the next subsection.

Conversely, to each vector field $v(x) = (v^j)$ in $\mathbb{R}^3$, one may associate a flow $\{\phi_t\}$ having v as its velocity field. [4] The map $\phi_t(p)$ with t a continuous parameter can be found by solving the system of ordinary differential equations, $\mathrm{d}x^j/\mathrm{d}t = v^j(x^1(t), x^2(t), x^3(t))$ with the initial condition, $x(0) = p$. Thus one finds an integral curve in a neighborhood of $t = 0$, which is a one-parameter family of maps $\phi_t(p)$ for any $p \in \mathbb{R}^3$, called a flow *generated* by the vector field v, where $v = \dot{x}_t = \dot{\phi}_t$. The map ϕ_t is

[3] According to the recent custom, we use the symbol := to define the left side by the right side, and =: to define the right side by the left side.

[4] The flow $\{\phi_t\}$ is considered to be diffeomorphisms of Sobolev class H^s in Section 7 ($s > n/2 + 1$ in $\mathbb{R}^n$, Appendix A.2).

a diffeomorphism, because $\phi_t(p)$ is differentiable, one-to-one, onto and ϕ_t^{-1} is differentiable, with respect to every point $p \in \mathbb{R}^3$. This is assured by the physical property that each fluid particle is a physical entity which keeps its identity during the motion, as long as two particles do not come to occupying an identical point simultaneously. [5]

Remark. Continuous distribution of fluid particles in a space has infinite degrees of freedom. Therefore, the velocity field of all the particles as a whole is regarded to be of infinite dimensional. In this context, the set of diffeomorphisms ϕ_t forms an infinite dimensional manifold $D^{(\infty)}$ and a point $\eta = \phi_t \in D^{(\infty)}$ represents a configuration (as a whole) of all particles at a given time t.

2.2.2. Tangent vector and differential operator

The vector fields we are going to consider on M^n are not an object residing in a flat Euclidean space. We need a sophisticated means to represent vectors which are different from a simple n-tuple of real numbers. In general on a manifold M^n, one can define a vector v tangent to the parameterized curve ϕ_t at any point x on the curve. We motivate the definition of vector as follows.

A *flow* $\phi_t(p) = (x_t^j(p))$ on an n-dimensional manifold M^n is described by the system of ordinary differential equations,

$$\frac{\mathrm{d}x_t^j}{\mathrm{d}t} = v^j(x_t^1, \ldots, x_t^n) \quad (j = 1, \ldots, n), \tag{5}$$

with the initial condition, $\phi_0 = p$. The one-to-one correspondence between the tangent vector $v = (v^j)$ to M^n at x and the first-order differential operator $\sum_j v^j(x)\partial/\partial x^j$, mediated by (5) and the n-dimensional version of (3), implies the following representation:

$$v(x) := \sum_j v^j(x) \frac{\partial}{\partial x^j}, \tag{6}$$

which defines the *vector field* $v(x)$ as a differential operator $v^j(x)\partial/\partial x^j$.

In fact, with a local coordinate patch (U, x_U) in the neighborhood of a point p, a curve will be described by n differentiable functions $(x_U^1(t), \ldots, x_U^n(t))$. The tangent vector at p is described by $v_U = (\dot{x}_U^1(0), \ldots, \dot{x}_U^n(0))$ where $\dot{x}(0) = \mathrm{d}x/\mathrm{d}t|_{t=0}$. If p also lies in the coordinate patch (V, x_V), then the same tangent vector is described by another n-tuple $v_V = (\dot{x}_V^1(0), \ldots, \dot{x}_V^n(0))$. In terms of the transformation function (1) on the overlapping domain which is now represented by $x_V^i = x_V^i(x_U^j)$, the two sets of tangent vectors are related by the chain rule,

$$v_V^i = \frac{\mathrm{d}x_V^i}{\mathrm{d}t}\bigg|_{t=0} = \sum_j \left(\frac{\partial x_V^i}{\partial x_U^j}\right) \frac{\mathrm{d}x_U^j}{\mathrm{d}t}\bigg|_{t=0} = \sum_j \left(\frac{\partial x_V^i}{\partial x_U^j}\right) v_U^j. \tag{7}$$

This suggests the transformation law of a tangent vector. This transformation defines the vector v of (6) that is independent of the local coordinate basis. In fact, by using the same transformation

[5] The present formulation is relevant to the times before a time of spontaneous formation of singularity (if any).

function, we obtain

$$v = \sum_j v_U^j(x) \frac{\partial}{\partial x_U^j} = \sum_j v_U^j(x) \sum_i \left(\frac{\partial x_V^i}{\partial x_U^j} \right) \frac{\partial}{\partial x_V^i} = \sum_i v_V^i(x) \frac{\partial}{\partial x_V^i}. \tag{8}$$

It is not difficult to see that the properties of the linear vector space are satisfied by the representation (6).[6] Usually, in the differential geometry, no distinction is made between a vector and its associated differential operator. The vector $v(x)$ thus defined at a point $x \in M^n$ is called a *tangent* vector.

2.2.3. Tangent space and tangent bundle

Each one of the n operators $\partial/\partial x^\alpha$ ($\alpha = 1, \ldots, n$) defines a vector, written as $\boldsymbol{\partial}/\boldsymbol{\partial} x^\alpha$, at each point x. The αth vector $\boldsymbol{\partial}/\boldsymbol{\partial} x^\alpha$ ($v^\alpha = 1$ and $v^j = 0$ for $j \neq \alpha$) is the tangent vector to the αth coordinate curve parameterized by x^α. The n vectors $\boldsymbol{\partial}/\boldsymbol{\partial} x^1, \ldots, \boldsymbol{\partial}/\boldsymbol{\partial} x^n$ form a basis of a vector space, and this base is called a *coordinate* basis. The basis vector $\boldsymbol{\partial}/\boldsymbol{\partial} x^\alpha$ is simply written as ∂_α (without using bold face font). A tangent vector X is written in general as[7]

$$X = X^j \partial_j \quad \text{or} \quad X_x = X^j(x) \partial_j.$$

The *tangent space* is defined by a vector space consisting of all tangent vectors to M^n at x and is written as $T_x M^n$. When the coefficients X^j are smooth functions $X^j(x)$ for $x \in M^n$, the $X(x)$ is called a *vector field* and an element of the *tangent bundle*, TM^n. Namely, the tangent bundle TM^n is defined as the collection of all tangent vectors at all points of M^n.

If $\boldsymbol{r} = (r^1, \ldots, r^N)$ is a position vector in the space $\mathbb{R}^N$ and M^n is a submanifold of $\mathbb{R}^N$: $M^n \subset \mathbb{R}^N$ ($n < N$), the vector $\boldsymbol{\partial}/\boldsymbol{\partial} x^\alpha$ is understood as $\partial_\alpha \equiv \boldsymbol{\partial}/\boldsymbol{\partial} x^\alpha = \partial \boldsymbol{r}/\partial x^\alpha = (\partial/\partial x^\alpha)(r^1, \ldots, r^N)$.[8]

2.2.4. Time-dependent velocity field

In the case of *time-dependent* velocity field, an additional coordinate x^0 is to be introduced, and the n equations (5) are replaced by the following $(n+1)$ equations:

$$\frac{\mathrm{d}x^j}{\mathrm{d}t} = X^j(x^0(t), x^1(t), \ldots, x^n(t)) \quad \text{with } X^0 = 1 \tag{9}$$

for $j = 0, 1, \ldots, n$. Naturally the additional equation is reduced to $x^0 = t$. Correspondingly, the tangent vector (i.e. the *velocity* vector) in the time-dependent case, denoted by the *hat* symbol, is written as

$$\hat{X} := \hat{X}^i \partial_i = X^0 \partial_0 + X^\alpha \partial_\alpha = \partial_t + X^\alpha \partial_\alpha, \tag{10}$$

where the index α denotes the spatial components ($\alpha = 1, \ldots, n$). *Non-velocity* tangent vector such as the Jacobi vector $\hat{J}$ is written simply as $\hat{J} = J^\alpha \partial_\alpha$ (see the footnote 17 of Section 7.3).

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[6] It is evident from (6) that the sum of two vectors at a point is again a vector at that point, and that the product of a vector by a real number is a vector.

[7] The summation convention is used hereafter, i.e. the summation with respect to j is understood for the pair of double indices like j.

[8] It is useful in later sections to keep in mind that the tangent space $T_x M^n$ is the usual n-dimensional *affine* subspace of $\mathbb{R}^N$.

2.3. Differential and inner product

2.3.1. Covector (1-form)

We regard the differential $\mathrm{d}f$ of a function f on M^n defined by $\mathrm{d}f[v] := vf$ as a linear functional $T_xM^n \to \mathbb{R}$ for any vector $v \in T_xM^n$. In local coordinates, we have $v = v^j \partial_j$. Using (6), we obtain

$$\mathrm{d}f[v] = \mathrm{d}f[v^j\partial_j] = vf = \sum_j v^j(x)\,\frac{\partial f}{\partial x^j}. \tag{11}$$

This is a basis-independent definition (see (8)). The differential $\mathrm{d}f[v^j\partial_j]$ is linear with respect to the scalar coefficient v^j. In particular, if f is the coordinate function x^i, we obtain, replacing f by x^i,

$$\mathrm{d}x^i[v] = \mathrm{d}x^i[v^j\partial_j] = v^j\,\mathrm{d}x^i\left[\frac{\partial}{\partial x^j}\right] = v^j\,\frac{\partial x^i}{\partial x^j} = v^j\delta^i_j = v^i.$$

The operator $\mathrm{d}x^i$ reads off the ith component of any vector v. It is seen that $\mathrm{d}x^i[\partial_j] = \delta^i_j$. Thus, the n functionals $\mathrm{d}x^i$ $(i = 1,\dots,n)$ yield the dual bases corresponding to the coordinate bases $(\partial_1,\dots,\partial_n)$ of a vector space T_xM^n. The dual bases $(\mathrm{d}x^1,\dots,\mathrm{d}x^n)$ form a dual space $(T_xM^n)^*$. The most general linear functional, $\alpha : T_xM^n \to \mathbb{R}$, is expressed in coordinates as

$$\alpha := a_1\,\mathrm{d}x^1 + \cdots + a_n\,\mathrm{d}x^n. \tag{12}$$

The α is called a *covector*, or *covariant vector*, or (differential) *one-form* (1-form), and is an element of the *cotangent* space $(T_xM^n)^*$. Corresponding to the covariant vector α, the vector v is also called a *contravariant vector*. Given a vector $v = v^j\partial_j$, the 1-form $\alpha[v]$ takes the value, $\alpha[v] = a_i\,\mathrm{d}x^i[v^j\partial_j] = a_iv^i$. When the coefficients a_i are smooth functions $a_i(x)$, the α is a 1-form *field* and an element of the *cotangent bundle*, $(TM^n)^*$.

The differential of a function f (without $[\cdot]$) is a typical example of the covector (1-form): $\mathrm{d}f = \sum_i(\partial f/\partial x^i)\,\mathrm{d}x^i$, where $\mathrm{d}x^i$ is a basis covector and $\partial f/\partial x^i$ is its component. This form holds in any manifold. In the next subsection, a vector $\mathrm{grad}\,f$ is defined as one corresponding to the covector $\mathrm{d}f$.

2.3.2. Inner (Scalar) product

Let the vector space T_xM^n be endowed with an inner (scalar) product $\langle\cdot,\cdot\rangle$. For each pair of vectors $X, Y \in T_xM^n$, the inner product $\langle X, Y\rangle$ is a real number, and it is bilinear and symmetric with respect to X and Y. Furthermore, the $\langle X, Y\rangle$ is *non-degenerate* in the sense that $\langle X, Y\rangle = 0$ for any Y only if $X = 0$. Writing $X = X^i\partial_i$ and $Y = Y^j\partial_j$, the inner product is given by

$$\langle X, Y\rangle := g_{ij}X^iY^j, \tag{13}$$

where $g_{ij} := \langle\partial_i, \partial_j\rangle = g_{ji}$ is the *metric tensor*. By definition, the inner product $\langle A, X\rangle$ is linear with respect to X when the vector A is fixed. Then the following operation on X, $\alpha[X] = \langle A, X\rangle$, is a linear functional. In other words, to each vector $A = A^j\partial_j$, one may associate a covector α. By definition, one has $\alpha[X] = g_{ij}A^jX^i = (g_{ij}A^j)X^i$. On the other hand, one has from (12), $\alpha[X] = a_i\,\mathrm{d}x^i[X] = a_iX^i$, in terms of the basis $\mathrm{d}x^i$. Thus one obtains

$$a_i = g_{ij}A^j =: A_i, \tag{14}$$

which defines A_i. Thus the component a_i is given by $g_{ij}A^j$ and written as A_i using the same letter A. The covector $\alpha = A_i\,\mathrm{d}x^i = (g_{ij}A^j)\,\mathrm{d}x^i$ is called the covariant version of the vector $A = A^j\partial_j$. In tensor analysis, one says that the upper index j is lowered by means of the metric tensor g_{ij} in (14). In other words, the covector is obtained by lowering the upper index of the vector by means of g_{ij}.

On the other hand, a vector A^j is obtained by raising the lower index of the covector A_i:

$$A^j = g^{ji}A_i, \tag{15}$$

which is equivalent to solving Eq. (14) to obtain A^j. This is verified by recalling that the metric tensor matrix $g = (g_{ij})$ is assumed non-degenerate, therefore that the inverse matrix g^{-1} must exist and is symmetric. The inverse is written as $g^{-1} =: (g^{ji})$ in the above equation using the same letter g. Thus we have the expression of the vector grad f as

$$(\mathrm{grad}\, f)^j = g^{ji}\,\frac{\partial f}{\partial x^i}. \tag{16}$$

2.4. Mapping of vectors and covectors

Let $\phi: M^n \to V^r$ be a smooth map. In addition, let us define the differential of the map ϕ by $\phi_*: T_xM^n \to T_yV^r$. In local coordinates, the map ϕ is represented by the function $y = F(x)$, where $x \in M^n$ and $y \in V^r$. Let $p(t)$ be a curve on M with $p(0) = p$ and $\dot{p}(0) = X$ (*tangent* vector), where $X \in T_pM^n$. Then, the differential map ϕ_* is defined by $Y = \phi_*X (= F_*X) := \mathrm{d}/\mathrm{d}t(F(p(t))|_{t=0}$. Thus, the vector X is mapped to Y, whose component is given as $Y^k = (\phi_*X)^k = (\partial F^k/\partial x^j)X^j = (\partial y^k/\partial x^j)X^j$, by definition. The transformation ϕ_* is linear with respect to the (scalar) coefficient X^j, and is also written as

$$\phi_*X = \phi_*\left[X^j\,\frac{\partial}{\partial x^j}\right] = X^j\phi_*\left[\frac{\partial}{\partial x^j}\right] = X^j\,\frac{\partial y^k}{\partial x^j}\,\frac{\partial}{\partial y^k} = Y^k\,\frac{\partial}{\partial y^k} = Y. \tag{17}$$

This is called a *push-forward* transformation of the velocity vector X to the vector Y (the velocity vector of the image curve at $F(p)$). The components of Y are given by

$$Y^k = \frac{\partial y^k}{\partial x^j}X^j. \tag{18}$$

This is also written as $Y = JX$, where $J = (J^k_j) = (\partial y^k/\partial x^j)$.

A manifold M^n is called a *submanifold* of a manifold V^r (where $n < r$) provided that there is a one-to-one smooth mapping $\phi: M^n \to V^r$ in which the matrix J has (maximal) rank n at each point. We refer to ϕ as an *imbedding* or an *injection*. This appears often when $V^r = \mathbb{R}^r$ so that we consider submanifolds of a Euclidean space $\mathbb{R}^r$.

Corresponding to the *push-forward* ϕ_*, one can define the *pull-back* ϕ^*, which is the linear transformation taking covectors at y to covectors at x, i.e. $\phi^*: (T_yV)^* \to (T_xM)^*$. Suppose that a vector X at $x \in M$ is transformed to $Y = \phi_*(X)$ at $y = \phi(x) \in V$, then the pull-back ϕ^* of a covector α (one-form) is defined, using the push-forward $\phi_*(X)$, by

$$(\phi^*\alpha)[X] := \alpha[\phi_*(X)] \tag{19}$$

for any one-form $\alpha = A_i\,\mathrm{d}y^i$. Below, we consider the pull-back of a function $f(x)$ (zero-form) which is given by $(\phi^* f)(x) = f(\phi(x))$. Note that, owing to $\mathrm{d}x^i[\partial_j] = \delta^i_j$, one has $\alpha\,[\partial/\partial y^k] = A_i\,\mathrm{d}y^i[\partial/\partial y^k] = A_k$. Writing as $\phi^*\alpha = a_i\,\mathrm{d}x^i$, one obtains $a_i = \phi^*\alpha[\partial/\partial x^i] = \alpha[\phi_*(\partial/\partial x^i)] = A_j\,\mathrm{d}y^j[\phi_*(\partial/\partial x^i)] = A_j(\partial y^j/\partial x^i)$.

Definition (19) characterizes a certain property of the pull-back transformation. That is, the value of the covector $\alpha = A_i\,\mathrm{d}y^i$ at the vector $Y = \phi_* X$ (in V) is equal to the value of the pull-back covector $\phi^*\alpha$ at the original vector X (in M). If one sets $A_i = \partial f/\partial y^i$, Eq. (19) expresses invariance of the differential: $\phi^*(\mathrm{d}f)_y = \phi^*[(\partial f/\partial y^i)\,\mathrm{d}y^i] = (\partial f/\partial y^i)(\partial y^i/\partial x^j)\,\mathrm{d}x^j = (\partial f/\partial x^j)\,\mathrm{d}x^j = (\mathrm{d}f)_x$. (See Section 2.6.2 for $M = V$.)

Based on this invariance, the general pull-back formula is defined for the integral of a form (covector) α over a curve σ as

$$\int_{\phi(\sigma)} \alpha = \int_\sigma \phi^*\alpha, \tag{20}$$

where $\phi : \sigma \subset M \to \phi(\sigma) \subset V$. In other words, the integral of a form α over the image $\phi(\sigma)$ is the integral of the pull-back $\phi^*\alpha$ over the original σ.

2.5. Lie group and invariant vector fields

A set G of smooth transformations (maps) of a manifold M into itself is called a group, provided that (i) with two maps $g, h \in G$, the product $gh = g \circ h$ belongs to $G : G \times G \to G$, (ii) for every $g \in G$, there is the inverse map $g^{-1} \in G$. From (i) and (ii), it follows that the group contains the identity map e (unity, identity): $gg^{-1} = g^{-1}g = e$.

A *Lie group* is a group which is a differentiable manifold, for which the operations (i) and (ii) are differentiable. A Lie group always has two families of diffeomorphisms, the left and right translations. With a fixed element $h \in G$,

$$L_h(g) = hg \quad (\text{or } R_h(g) = gh) \quad \text{for any } g \in G,$$

where L_h (or R_h) indicates the *left* (or *right*) translation of the group onto itself, respectively. Note that $L_g(h) = R_h(g) = gh$. The operation inverse to L_h (or R_h) is simply $L_{h^{-1}}$ (or $R_{h^{-1}}$), respectively.

It is said that a vector field X^{R}, or X^{L} on G is *right-invariant*, or *left-invariant* respectively, if it is invariant under all right-translations, or left-translations, that is, for all $g, h \in G$,

$$(R_h)_* X^{\mathrm{R}}_g = X^{\mathrm{R}}_{gh} \quad \text{or} \quad (L_h)_* X^{\mathrm{L}}_g = X^{\mathrm{L}}_{hg}. \tag{21}$$

Given a tangent vector X_e to G at the identity e, one may right-translate or left-translate X_e to every point $g \in G$ as

$$X^{\mathrm{R}}_g = (R_g)_* X_e = X_e \circ g, \tag{22}$$

$$X^{\mathrm{L}}_g = (L_g)_* X_e = g_* X_e, \tag{23}$$

respectively. It should be readily seen from (22) that $(R_h)_* X^{\mathrm{R}}_g = X^{\mathrm{R}}_{gh}$, hence transformation (22) gives a right-invariant field generated by X_e. Similarly, transformation (23) gives a left-invariant field. The right-translation of $X_e(x)$ by g, $X_e \circ g(x)$, is understood as follows: the map g acts to the point x first and then the vector X_e is taken at the point $g(x)$.

Suppose that g_t is a curve on G described in terms of a parameter t. The left transformation of g_t by $g_{\Delta t}$ for an infinitesimal Δt is given by $g_{\Delta t} \circ g_t$. Hence the t-derivative is expressed as $\dot{g}_t = v \circ g_t$, where $v = \lim_{\Delta t \to 0} (g_{\Delta t} - \mathrm{id})/\Delta t$, where id denotes the identity map. Thus, the *left*-transformation leads to the *right*-invariant vector field. Similarly, the *right*-transformation leads to the *left*-invariant vector field.

For two right-invariant tangent vectors X_g^{R} and Y_g^{R}, the metric (13) is called *right-invariant* if

$$\langle X_g^{\mathrm{R}}, Y_g^{\mathrm{R}} \rangle = \langle X_e, Y_e \rangle.$$

Similarly, the metric is *left-invariant* if $\langle X_g^{\mathrm{L}}, Y_g^{\mathrm{L}} \rangle = \langle X_e, Y_e \rangle$ for left-invariant tangents, $X_g^{\mathrm{L}}, Y_g^{\mathrm{L}}$. Examples of the left-invariant field are seen in Sections 4 and 8.

2.6. Lie algebra and Lie derivative

2.6.1. Adjoint operator and Lie bracket

Every pair of vector fields defines a new vector field called the *Lie bracket* $[\cdot,\cdot]$. More precisely, the tangent space T_eG at the identity e of a Lie group G, equipped with the bracket operation $[\cdot,\cdot]$ of bilinear skew-symmetric pairing, is called the *Lie algebra* g of the group G, where $[\cdot,\cdot] : \mathrm{g} \times \mathrm{g} \to \mathrm{g}$, if the bracket satisfies the Jacobi identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \tag{24}$$

for any triplet of $X, Y, Z \in \mathrm{g} = T_eG$. The $[\cdot,\cdot]$: is called the *Lie bracket*.

A Lie group G acts as a group of linear transformations on its own Lie algebra g, called the *adjoint group* $\mathrm{Ad}(G)$. That is, for $\xi \in G$, there is an $\mathrm{Ad}_\xi \in \mathrm{Ad}(G)$, such that

$$\mathrm{Ad}_\xi Y := (L_\xi)_* \circ (R_{\xi^{-1}})_* Y = \xi Y \xi^{-1},$$

for all $Y \in \mathrm{g}$. The operator Ad_ξ is a linear transformation on g, transforming $Y \in \mathrm{g}$ into $\mathrm{Ad}_\xi Y \in \mathrm{g}$, and $\mathrm{Ad}(G)$ is called the *adjoint* representation of G, acting in the Lie algebra space.

Consider a curve $\xi_t : t \to G$ with the tangent $\dot{\xi}_0 = X$. Using the inverse $(\xi_t)^{-1} = \xi_t^{-1}$, the adjoint transformation $\mathrm{Ad}_{\xi_t^{-1}} Y$ is a function of t. Its derivative with respect to t is a linear transformation from Y to $\mathrm{ad}_X Y$ defined by

$$\mathrm{ad}_X Y := \frac{\mathrm{d}}{\mathrm{d}t} \xi_t^{-1} Y \xi_t \bigg|_{t=0} = [X, Y]. \tag{25}$$

This defines the Lie bracket $[X, Y]$.[9] However, its explicit expression depends on each group or each dynamical system considered. It can be shown that the bracket $[X, Y]$ thus introduced satisfies all the properties required for the Lie bracket in each example considered below. The bracket operation is usually called the *commutator*. The ad_X is a linear transformation from $\mathrm{g} \to \mathrm{g}$, by the representation, $Y \to [X, Y]$. The operator ad_X stands for the image of an element X under the action of the linear map ad.

[9] Most textbooks in mathematics adopt this definition. Arnold (1966, 1978) uses the definition of its opposite sign which is convenient for the physical systems related to rotation groups (see (26)) like those of Sections 4 and 8, characterized with the left-invariant metric. In fact, the right-invariant vector field is represented by $X \circ g = (\mathrm{d}/\mathrm{d}t)(\xi_t \circ g)$ for $g \in G$, whereas the left-invariant field is given by $g_* X = (\mathrm{d}/\mathrm{d}t)(g \circ \xi_t)$. This difference results in the different signs of the corresponding Lie brackets of the right- and left-invariant fields (see de Azcárraga and Izquierdo, 1995).

As a first example, consider the rotation group $G = \mathrm{SO}(3)$, of which an element A is represented by a 3×3 orthogonal matrix of $\det A = 1$. Let $\xi(t)$ be a curve issuing from e with the velocity $\mathbf{a}$ on SO(3). Then one has $\xi(t) = e + t\mathbf{a} + O(t^2)$ for an infinitesimal time parameter t, where $\mathbf{a} = \dot{\xi}(0)$ is an element of the algebra g (*usually written as* $\mathbf{so}(3)$ *using lower case letters of bold face*) and represented by a skew-symmetric matrix due to the orthogonality (with the restriction $\det \xi(t) = 1$). The algebra element $\mathbf{a}$ is called a generator of $\xi(t)$. Then, for $\mathbf{a}, {}^\forall\mathbf{b} \in \mathbf{so}(3)$, the operator $\mathrm{ad}_{\mathbf{a}} : \mathrm{g} \to \mathrm{g}$ is given by

$$\mathrm{ad}_{\mathbf{a}}\mathbf{b} = [\mathbf{a},\mathbf{b}] = -(\mathbf{ab} - \mathbf{ba}) =: -\mathbf{c}, \tag{26}$$

where the minus sign in front of $\mathbf{ab} - \mathbf{ba}$ is due to definition (25) and verified just below. The commutator $\mathbf{ab} - \mathbf{ba} = \mathbf{c}$ is in the form of another skew-symmetric matrix. The well-known representation of a 3×3 skew-symmetric matrix $\mathbf{a}$ in terms of a three-component (axial) vector $\hat{\mathbf{a}}$ leads to the rule of cross product,

$$\hat{\mathbf{c}} = \hat{\mathbf{a}} \times \hat{\mathbf{b}}. \tag{27}$$

To show (26), let $s \mapsto \eta(s)$ be another curve with the initial velocity $\dot{\eta}(0) = \mathbf{b}$. Then,

$$\begin{aligned}\xi(t)^{-1}\eta(s)\xi(t) &= (e - t\mathbf{a} + O(t^2))(e + s\mathbf{b} + O(s^2))(e + t\mathbf{a} + O(t^2)) \\ &= e + s(\mathbf{b} - t(\mathbf{ab} - \mathbf{ba}) + O(t^2)) + O(s^2)\end{aligned} \tag{28}$$

as $t, s \to 0$. The differentiation $\partial^2/\partial t\partial s$ leads to Eq. (26). As far as the time evolution is concerned, we define the bracket for the left-invariant system such as the rotation group as

$$[\mathbf{a},\mathbf{b}]^{(\mathrm{L})} := -[\mathbf{a},\mathbf{b}] = (\mathbf{ab} - \mathbf{ba}) = \mathbf{c}. \tag{29}$$

2.6.2. Lie derivative and Lagrange derivative

Suppose that we are given a vector field $X = X^i\partial_i$ on a manifold M^n. As described in Section 2.2, with every such vector field, we associate a *flow*, or one-parameter group of *diffeomorphisms* $\xi_t : M^n \to M^n$, for which $\xi_0 = \mathrm{id}$ [10] and $(\mathrm{d}/\mathrm{d}t)\xi_t x|_{t=0} = X(x)$. A first-order differential operator $\mathscr{L}_X$ on a scalar function $f(x)$ on M is defined by

$$\mathscr{L}_X f(x) := \left.\frac{\mathrm{d}}{\mathrm{d}t}(\xi_t)^* f(x)\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t} f(\xi_t x)\right|_{t=0} = X^i \frac{\partial}{\partial x^i} f(x) \tag{30}$$

(see (19) and below it, and (3)). The operator $\mathscr{L}_X$ is called the *Lie derivative*. The derivative $\mathscr{L}_X f$ of a function f (a zero-form) is defined by the time derivative of its pull-back $\xi_t^* f$ at $t = 0$. The point $\xi_t x$ moves forward in accordance with the flow of velocity X. Relatively observing, the pull-back $\xi_t^* f$ is estimated at x, and its time derivative defines the Lie derivative. This is sometimes called as a *derivative of a fisherman* (Arnold and Khesin, 1998) sitting at a fixed place x. [11]

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[10] The id may be also written as e. The id is used here in order to emphasize that this is a *map*.

[11] The Lie derivative $\mathscr{L}_X$ also acts on any form fields α in the same way, $(\mathrm{d}/\mathrm{d}t)(\xi_t)^*\alpha$ like (30). On the contrary, to a vector field Y, the Lie derivative is defined by (31) in terms of the push-forward $(\xi_t)_*$. This definition is different from that of Arnold (1978, 1966) by the sign, but the same as the present definition of $\mathrm{ad}_X Y$ of (25).

In the fluid dynamics however, the same derivative is called the *Lagrange* derivative, which refers to the third and fourth expressions,

$$\frac{\mathrm{D}f}{\mathrm{D}t} := \frac{\mathrm{d}}{\mathrm{d}t} f(\xi_t x) = X^i \frac{\partial}{\partial x^i} f(x).$$

This is understood as denoting the time derivative observed by the fluid particle at $\xi_t x$ moving with the flow.

Now, let us consider the Lie derivative of the vector field $Y(x)$. To that end, suppose that we are given another vector field $Y(x) = Y^i \partial_i$, and denote the flow generated by $Y(x)$ as η_s with $\eta_0 = \mathrm{id}$. The flow is transporting the vector $Y(x)$ in front of a fisherman sitting at a point x. After an infinitesimal time t, the fluid particle at x will arrive at $\xi_t x$. We take the vector field Y at this point $\xi_t x$ and translate it backwards to the original point x by the inverse map of the push-forward, that is $(\xi_t)^{-1} Y(\xi_t x)$, in precise $(\xi_t)_*^{-1} Y(\xi_t x)$. Its time derivative is the Lie derivative, given by

$$\mathscr{L}_X Y := \lim_{t\to 0} \frac{\xi_{t*}^{-1} Y(\xi_t x) - Y(x)}{t} = \lim_{t\to 0} \xi_{t*}^{-1} \left. \frac{Y\xi_t - \xi_{t*} Y}{t} \right|_x = \lim_{t\to 0} \left. \frac{Y\xi_t - \xi_{t*} Y}{t} \right|_x . \tag{31}$$

The definition expression is nothing but that of $\mathrm{ad}_X Y(x)$ according to (25).

Thus we have

$$\mathscr{L}_X Y = \frac{\mathrm{d}}{\mathrm{d}t} \xi_{t*}^{-1} Y \xi_t \Big|_{t=0} = \frac{\partial}{\partial t} \frac{\partial}{\partial s} \xi_t^{-1} \eta_s \xi_t \Big|_{t=0,s=0} \tag{32}$$

$$= \mathrm{ad}_X Y =: [X, Y], \tag{33}$$

where $[X, Y]$ is the *Lie bracket*. The two diffeomorphisms ξ_t and η_s corresponding to the vector fields X and Y respectively can be written in the form:

$$\xi_t: \quad x \mapsto x + tX(x) + O(t^2), \quad t \to 0, \tag{34}$$

$$\eta_s: \quad x \mapsto x + sY(x) + O(s^2), \quad s \to 0. \tag{35}$$

Recalling that $\eta_s \xi_t x$ is given by $\eta_s(\xi_t(x))$ by the composition rule, we have

$$\begin{aligned} \eta_s \xi_t: \quad x \mapsto & x + tX(x) + O(t^2) + sY(x + tX(x) + O(t^2)) + O(s^2) \\ & = x + tX(x) + s(Y(x) + tX^j \partial_j Y + O(t^2)) + O(t^2, s^2). \end{aligned}$$

The inverse ξ_t^{-1} is given by $x \mapsto x - tX(x) + O(t^2)$. Finally, we have (Arnold and Khesin, 1998)

$$\xi_t^{-1} \eta_s \xi_t: x \mapsto x + sY(x) + st \left(X^j \frac{\partial Y}{\partial x^j} - Y^j \frac{\partial X}{\partial x^j} \right) + O(t^2, s^2). \tag{36}$$

Thus, according to definition (32), it is found that the Lie derivative of the vector field Y with respect to X is given by

$$\mathscr{L}_X Y = [X, Y] = \{X, Y\} = \{X, Y\}^k \partial_k, \tag{37}$$

$$\{X, Y\}^k := X^j \frac{\partial Y^k}{\partial x^j} - Y^j, \frac{\partial X^k}{\partial x^j}, \tag{38}$$

where $\{X, Y\}$ is the *Poisson bracket*, also represented as (using $\mathscr{L}_X = X^i \partial_i$),

$$\{X, Y\} = \mathscr{L}_X \mathscr{L}_Y - \mathscr{L}_Y \mathscr{L}_X = [\mathscr{L}_X, \mathscr{L}_Y] = \mathscr{L}_{\{X,Y\}}. \tag{39}$$

If $\xi_t \eta_s = \eta_s \xi_t$, then obviously we have $[X, Y] = 0$. This suggests that the coordinate bases commute. In fact, for $X = \partial_\alpha$, $Y = \partial_\beta$, we obtain from (37)

$$[\partial_\alpha, \partial_\beta] = \partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha = 0. \tag{40}$$

In general,

$$\xi_t \eta_s - \eta_s \xi_t = [X, Y]st + O(t^2, s^2).$$

If $[X, Y] = 0$, then we have $\xi_t \eta_s - \eta_s \xi_t = O(t^2, s^2)$.

In the fluid dynamics, the Lagrange derivative of the vector $Y = Y^k \partial_k$ is defined by the following right-hand side,

$$\frac{\mathrm{D}}{\mathrm{D}t} Y = \frac{\mathrm{D}}{\mathrm{D}t} Y^k(\xi_t x) \partial_k = X^j(x) \frac{\partial Y^k(x)}{\partial x^j} \partial_k, \tag{41}$$

which corresponds to only the first term on the right-hand side of (37).

Remark. A vector field Y defined along the integral curve ξ_t generated by the tangent field X is said to be *invariant* if $Y(\xi_t x) = (\xi_t)_* Y(x)$. Substituting this to the last expression of (31), it is readily seen that $\mathscr{L}_X Y = 0$, or rewriting it,

$$\mathscr{L}_X Y = \left(\frac{\mathrm{D}Y^i}{\mathrm{D}t} - Y^j \frac{\partial X^i}{\partial x^j} \right) \partial_i = 0. \tag{42}$$

In the fluid dynamics, the equation $\mathscr{L}_X Y = 0$ is called the equation of *frozen field.*[12] If we set $\phi = \xi_t$ in (17) together with $X = Y(x)$ and $Y = Y(\xi_t x)$, then the equation $Y(\xi_t x) = (\xi_t)_* Y(x)$ represents the push-forward transformation. Therefore, the solution of (42) is given by Eq. (18), $Y^\alpha(t) = Y^j(0) \partial y^\alpha / \partial x^j$, which is called the *Cauchy solution* in *fluid dynamics*.

2.7. Diffeomorphism of a circle S^1

Diffeomorphism of the manifold S^1 (circle of periodicity T) is represented by a map $g : x \in S^1 \to g(x) \in S^1$ and $g(x + T) = g(x)$. Such maps constitute a group $D(S^1)$ of diffeomorphisms with the composition law:

$$h = g \circ f, \quad \text{i.e.} \quad h(x) = g(f(x)) \in D(S^1)$$

for $f, g \in D(S^1)$. The diffeomorphism is a map of infinite degrees of freedom, i.e. having *pointwise* degrees of freedom. In Section 5, the diffeomorphism is assumed to be orientation-preserving in the sense that $g'(x) > 0$, where the prime denotes $\partial/\partial x$.

Associated with a flow $\xi_t(x)$ that is a smooth sequence of diffeomorphisms with the time parameter t (see (34)), its tangent field at ξ_t is defined by

$$\dot{\xi}_t(x) := \frac{\mathrm{d}}{\mathrm{d}t} \xi_t(x) \bigg|_t = \lim_{\tau \to 0} \frac{\xi_\tau(x) - \mathrm{id}}{\tau} \circ \xi_t(x) = u(x) \circ \xi_t(x),$$

[12] The Jacobi field Y ($=J$ below) satisfies this equation (see Section 7.3.4).

in a right-invariant form. The tangent field at the identity (id) is given by $X(x)=u(x)=\mathrm{d}\xi_t(x)/\mathrm{d}t|_{t=0}$. Put it in another way, an element X in the *tangent space* $T_{\mathrm{id}}D(S^1)$ at id is represented as $X = u(x)\partial_x \in TS^1$, where TS^1 is the tangent bundle over the manifold S^1. For the two diffeomorphisms ξ_t and η_t corresponding to the vector fields X and Y respectively, the *Lie bracket* (commutator) is given by (37) and (38):

$$[X, Y] = (uv' - vu')\partial_x \tag{43}$$

for $X = u(x)\partial_x,\ Y = v(x)\partial_x \in TS^1$. This Lie algebra is sometimes called *Witt algebra* (de Azcárraga and Izquierdo, 1995).

3. Riemannian geometry

3.1. Riemannian metric

On a Riemannian manifold M^n, a positive definite inner product $\langle\cdot,\cdot\rangle$ is defined on the tangent space T_xM^n at $x = (x^1,\dots,x^n)\in M$ with a differentiable fashion. For two tangent fields $X = X^i(x)\partial_i,\ Y = Y^j(x)\partial_j \in TM^n$ (the tangent bundle, see Section 2.2.3), the *Riemannian metric* is given by

$$\langle X, Y\rangle(x) = g_{ij}X^i(x)Y^j(x),$$

as already defined in (13),[13] where the metric tensor, $g_{ij}(x) = \langle\partial_i, \partial_j\rangle = g_{ji}(x)$, is symmetric and differentiable with respect to x^i. This bilinear quadratic form is called the *first fundamental form*, which is represented as $\mathrm{I} := g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ where $\mathrm{d}x^i$ is a one-form. Using $X = X^i(x)\partial_i,\ Y = Y^j(x)\partial_j$, we have

$$\mathrm{I}(X, Y) = g_{ij}\,\mathrm{d}x^i(X)\,\mathrm{d}x^j(Y) = g_{ij}X^iY^j. \tag{44}$$

The inner product is *non-degenerate*, if

$$\langle X, Y\rangle = 0 \quad \text{for all } Y \in TM^n,\ \text{only when } X = 0. \tag{45}$$

3.2. Examples of metric tensor

3.2.1. Finite dimensions

Consider a dynamical system of N degrees of freedom in a *gravitational* field with the potential $V(\bar q)$ and the kinetic energy

$$T = \frac{1}{2}a_{ij}\dot q^i\dot q^j \quad \text{where } \bar q = (q^i) \quad (i = 1,\dots,N).$$

The metric is defined by $g_{ij}\dot q^i\dot q^j$, where

$$g_{ij} = g^J_{ij}(\bar q) = 2(E - V(\bar q))a_{ij} \quad \text{for } i, j = 1,\dots,N$$

[13] If the inner product is only non-degenerate rather than positive definite, the resulting structure on M^n is called a *pseudo*-Riemannian.

is the *Jacobi's* metric tensor (Pettini, 1993). In Section 6, we consider a physical system in terms of another metric called the Eisenhart metric g^E_{ij}.

3.2.2. Infinite dimensions

Metric on the group $D(S^1)$ of diffeomorphisms (Section 2.7) is defined for the right-invariant tangent fields $U_\xi(x) = u \circ \xi(x)$ and $V_\xi(x) = v \circ \xi(x)$ in the following invariant way (also see Section 7.2):

$$\langle U, V\rangle_\xi := \int_{S^1} (U_\xi \circ \xi^{-1}, V_\xi \circ \xi^{-1})_x \, \mathrm{d}x = \int_{S^1} u(x)v(x)\,\mathrm{d}x =: \langle X, Y\rangle \tag{46}$$

for $X = u(x)\partial_x$, $Y = v(x)\partial_x \in T_{\mathrm{id}}D(S^1)$, where $(\cdot,\cdot)_x$ denotes the pointwise inner product at $x \in S^1$. Thus we have $\langle U, V\rangle_\xi = \langle U, V\rangle_{\mathrm{id}} = \langle X, Y\rangle$.

3.3. Covariant derivative (connection)

Let M^2 be a surface in $\mathbb{R}^3$. Consider a vector field $Y(x)$ that is tangent to M^2 and defined along a parameterized curve $p(t) \in M^2$. From the ordinary derivative $\mathrm{d}Y/\mathrm{d}t$ in $\mathbb{R}^3$, its tangential part (tangent to M^2) is obtained by projection and denoted as $\nabla Y/\mathrm{d}t$ called the *intrinsic* derivative. Choose any curve on M^2 through p whose tangent at p is the vector U. We then define the *covariant derivative* $\nabla_U Y$ of a vector field Y in the direction of the vector U to be the intrinsic derivative: $\nabla_U Y := \nabla Y/\mathrm{d}t$. Parallel displacement of a vector Y along the curve is defined by $\nabla_U Y = 0$.

In general, on a manifold M^n, consider vector fields X, Y defined near p and vectors U and V defined at $p \in M^n$. A covariant derivative is an operator ∇, that assigns a vector $\nabla_U X$ at p to each pair of X and U, that satisfies the following relations:

$$\begin{aligned}
&\text{(i)} \quad \nabla_U(aX + bY) = a\nabla_U X + b\nabla_U Y,\\
&\text{(ii)} \quad \nabla_{aU+bV} X = a\nabla_U X + b\nabla_V X,\\
&\text{(iii)} \quad \nabla_U(f(x)X) = f(x)\nabla_U X + (Uf)X
\end{aligned} \tag{47}$$

for a smooth function $f(x)$ and a, $b \in \mathbb{R}$, where $Uf = \mathrm{d}f[U]$. Using the representations $X = X^i\partial_i$ and $Y = Y^j\partial_j$ and applying the above properties (i)–(iii), we obtain

$$\nabla_X Y = \nabla_{X^i\partial_i}(Y^j\partial_j) = X^i\nabla_{\partial_i}(Y^j\partial_j) = (X^i\partial_i Y^j)\partial_j + X^iY^j\Gamma^k_{ij}\partial_k, \tag{48}$$

$$\nabla_{\partial_i}\partial_j =: \Gamma^k_{ij}\partial_k, \tag{49}$$

where Γ^k_{ij} is called the Christoffel symbols. Therefore,

$$(\nabla_X Y)^i = X^j\frac{\partial Y^i}{\partial x^j} + \Gamma^i_{jk}X^jY^k = \mathrm{d}Y^i(X^j\partial_j) + (\Gamma^i_{jk}Y^k)X^j \tag{50}$$

$$:= \nabla Y^i(X), \quad \nabla Y^i = \mathrm{d}Y^i + \Gamma^i_{jk}Y^k\,\mathrm{d}x^j, \tag{51}$$

where ∇Y^i is a connection one-form. In this regard, the operation $\nabla_X Y$ is also called an *affine connection*. On a manifold M^n, an *affine frame* on M^n consists of n vector fields $e_k = \partial_k$ $(k = 1,\dots,n)$, which are linearly independent and furnish a basis of the tangent space at each point p. Then the symbols Γ^i_{jk} defined by $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$ are called the coefficients of the affine connection.

Most dynamical systems are *time-dependent* and every tangent vector is written in the form, $\hat{X} = \hat{X}^i \partial_i = \partial_t + X^\alpha \partial_\alpha$, where $x^0 = t$ and α denotes the indices of the spatial part, $\alpha = 1, \ldots, n$ (see (10)). Correspondingly, the connection is written as

$$\nabla_{\hat{X}} \hat{Y} = \nabla_{\hat{X}^i \partial_i} \hat{Y}^j \partial_j = \nabla_{\partial_t} \hat{Y}^j \partial_j + X^\alpha \nabla_{\partial_\alpha} (\hat{Y}^j \partial_j),$$

where $\hat{Y} = \partial_t + Y^\alpha \partial_\alpha$. Writing the spatial part of $\hat{X}$ and $\hat{Y}$ as $X = X^\alpha \partial_\alpha$ and $Y = Y^\alpha \partial_\alpha$ respectively, we assume

$$\nabla_{\partial_t} \partial_k = 0 \quad \text{implying } \Gamma^i_{0k} = 0 \quad (i, k = 0, \ldots, n). \tag{52}$$

Thus, we obtain

$$\nabla_{\partial_t} Y = \partial_t Y = \frac{\partial Y^\alpha}{\partial t} \partial_\alpha, \tag{53}$$

$$\nabla_{\hat{X}} Y = \partial_t Y + \nabla_X Y. \tag{54}$$

Applying the symmetry associated with the Riemannian connection described in the next section, we have $\Gamma^i_{k0} = \Gamma^i_{0k} = 0$, suggesting that the time axis x^0 is straight.

3.4. Riemannian connection

There is a unique connection ∇ on a Riemannian manifold M called *Riemannian* connection or *Levi–Civita* connection that satisfies

$$\text{(i)} \quad \nabla_X Y - \nabla_Y X = [X, Y] \quad \text{(torsion free)}, \tag{55}$$

$$\text{(ii)} \quad Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \quad \text{(compatibility with metric)} \tag{56}$$

for vector fields X, Y and Z, where $Z\langle\cdot,\cdot\rangle = Z^j \partial_j \langle\cdot,\cdot\rangle$. The torsion-free requires the symmetry, $\Gamma^k_{ij} = \Gamma^k_{ji}$, with respect to i and j. One consequence of the second compatibility condition with metric will be given at the end of Section 3.5 (Frankel, 1997; Milnor, 1963).

Owing to the two properties, the Riemannian connection satisfies the following identity:

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = {} & X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & + \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle. \end{aligned} \tag{57}$$

This equation defines the connection ∇ by means of the inner product $\langle\cdot,\cdot\rangle$ and the commutator $[\cdot,\cdot]$.

In most dynamical systems studied below, the metrics are defined to be invariant (with respect to either right or left translation). Thus, if X, Y, Z are invariant vector fields, the first three terms on the right-hand side of (57) vanish identically. Hence on the Riemannian manifold of *invariant vector fields*, Eq. (57) reduces to

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle. \tag{58}$$

3.5. Covariant derivative along a parameterized curve

Consider a curve $x(t)$ on M^n passing through p whose tangent at p is given by $T = \mathrm{d}x/\mathrm{d}t = \dot{x}$, and let Y be a tangent vector field defined along the curve $x(t)$. According to (50), the covariant derivative $\nabla_T Y$ is also written as

$$\nabla_T Y = \frac{\boldsymbol{\nabla} Y}{\mathrm{d}t} = \left[\frac{\partial Y^i}{\partial x^k} + \Gamma^i_{kj} Y^j\right] \dot{x}^k \partial_i = \left[\frac{\mathrm{d}}{\mathrm{d}t} Y^i + \Gamma^i_{kj} \dot{x}^k Y^j\right] \partial_i. \tag{59}$$

The second expression $\boldsymbol{\nabla} Y/\mathrm{d}t$ emphasizes the derivative along the curve $x(t)$ parameterized with t. On the manifold M^n endowed with the connection $\boldsymbol{\nabla}$, one can consider *parallel* displacement of a vector Y along a parameterized curve $x(t)$, which is defined by vanishing covariant derivative: $\boldsymbol{\nabla} Y/\mathrm{d}t = \nabla_T Y = 0$. For the vector fields translated parallel along the curve, we have

$$\langle X, Y \rangle = \text{constant} \quad (\text{under parallel translation}),$$

because from (56) the scalar product is invariant, i.e. $T\langle X, Y \rangle = 0$. For another interpretation of the covariant derivative, see Section 3.8.1.

3.6. Curvature tensors

The curvature transformation $R(X, Y)$ for a pair of vector fields $X,\ Y \in TM$ is defined by

$$R(X, Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \tag{60}$$

$$= (R^\alpha_{kij} Z^k X^i Y^j)\partial_\alpha, \tag{61}$$

$$R^\alpha_{kij} := \partial_i \Gamma^\alpha_{jk} - \partial_j \Gamma^\alpha_{ik} + \Gamma^\beta_{jk}\Gamma^\alpha_{i\beta} - \Gamma^\beta_{ik}\Gamma^\alpha_{j\beta}, \tag{62}$$

which describes a linear transformation $T_xM \to T_xM$, i.e. $Z \to R(X, Y)Z$, as seen in (61), where R^α_{kij} is the *Riemannian curvature tensor*. This is obtained by using definition (48) for the covariant derivative $\nabla_X Y$ repeatedly and using the form $[X, Y] = \{X, Y\}^k \partial_k$ (see (37), (38)) for $\nabla_{[X,Y]}$. All the derivative terms of X^i, Y^j, Z^k cancel out and only the non-derivative terms remain, resulting in expressions (61) with definition (62). Expression (62) can be derived compactly as follows. Using the particular representation, $X = \partial_i$, $\ Y = \partial_j$ and $Z = \partial_k$, we have $R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i}(\nabla_{\partial_j}\partial_k) - \nabla_{\partial_j}(\nabla_{\partial_i}\partial_k) = R^\alpha_{kij}\partial_\alpha$, where the third term $\nabla_{[\partial_i, \partial_j]}$ does not appear because $[\partial_i, \partial_j] = 0$. Thus the definitive equation $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$ leads to representation (62), defining R^α_{kij} in terms of Γ^k_{ij} only. From the definition, one may write $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, which clearly shows the anti-symmetry $R(X, Y) = -R(Y, X)$. Correspondingly, we have $R^\alpha_{kij} = -R^\alpha_{kji}$. Taking the inner product of $R(X, Y)Z$ with $W = W^\alpha\partial_\alpha \in TM$, we have

$$\langle W, R(X, Y)Z \rangle = R^m_{\beta ij}\langle \partial_m, \partial_\alpha \rangle W^\alpha Z^\beta X^i Y^j = R_{\alpha\beta ij} W^\alpha Z^\beta X^i Y^j, \tag{63}$$

where $R_{\alpha\beta ij} = g_{m\alpha} R^m_{\beta ij}$ and $g_{m\alpha} = \langle \partial_m, \partial_\alpha \rangle$.

In addition to the anti-symmetry $\langle W, R(X, Y)Z \rangle + \langle W, R(Y, X)Z \rangle = 0$, one can show the following anti-symmetry (Milnor, 1963),

$$\langle W, R(X, Y)Z \rangle + \langle Z, R(X, Y)W \rangle = 0. \tag{64}$$

In fact, using (56) repeatedly, we obtain

$$\langle W, \nabla_X \nabla_Y Z\rangle = -\langle \nabla_X \nabla_Y W, Z\rangle + XY\langle W, Z\rangle - \langle \nabla_X W, \nabla_Y Z\rangle - \langle \nabla_Y W, \nabla_X Z\rangle$$

and similar expression for $\langle W, \nabla_Y \nabla_X Z\rangle$. Noting that $(XY - YX)\langle W, Z\rangle = [X, Y]\langle W, Z\rangle = \langle \nabla_{[X,Y]} W, Z\rangle + \langle W, \nabla_{[X,Y]} Z\rangle$, we obtain (64).

The Christoffel symbol Γ^k_{ij} is in turn represented in terms of the metric tensor $g = (g_{ij})$ by the following formula:

$$\Gamma^k_{ij} = \frac{1}{2} g^{km}(\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij}), \tag{65}$$

where g^{km} denotes the component of the inverse g^{-1}, that is $g^{km} = (g^{-1})^{km}$, satisfying the relations $g^{km} g_{ml} = g_{lm} g^{mk} = \delta^k_l$ (Kronecker's delta). Note that $\Gamma^k_{ij} = \Gamma^k_{ji}$ since $g_{ij} = g_{ji}$. Formula (65) can be verified by using (56), $g_{ij}(x) = \langle \partial_i, \partial_j\rangle$ and $\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$, and noting that

$$\partial_m g_{ij} = \partial_m \langle \partial_i, \partial_j\rangle = \langle \nabla_{\partial_m} \partial_i, \partial_j\rangle + \langle \partial_i, \nabla_{\partial_m} \partial_j\rangle = \Gamma^k_{mi} g_{kj} + \Gamma^k_{mj} g_{ki}$$

and that $\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij} = 2 g_{km} \Gamma^k_{ij}$.

3.7. Induced connection and second fundamental form

Let V^r be a submanifold of a Riemannian manifold M^n with the metric g_{ij}. Let us consider the restriction of the Riemannian metric g_{ij} to the tangent vectors to V^r. This induces the Riemannian metric (an induced metric) for V^r. An arbitrary vector field Z of M^n defined along V^r can be decomposed into two orthogonal components: $Z = Z_V + Z_N$, where $Z_V = \mathtt{P}\{Z\}$ is the projected component to V^r and $Z_N = \mathtt{Q}\{Z\}$ the component perpendicular to V^r (see Appendix A.2 for the Helmholtz decomposition of vector fields). The symbols $\mathtt{P}$ and $\mathtt{Q}$ denote the orthogonal projections onto the space V^r and the space orthogonal to it. Let ∇^M be the Riemannian connection for M^n, and define a new connection ∇^V for V^r $(r < n)$ as follows. Consider a vector X tangent to V^r and a vector field Z in M^n defined along V^r where Z is not necessarily tangent to V^r. Then the ∇^V is defined by

$$\nabla^V_X Z := \nabla^M_X Z - \mathtt{Q}\{\nabla^M_X Z\} = \mathtt{P}\{\nabla^M_X Z\}, \tag{66}$$

where the right-hand side is the projection of $\nabla^M_X Z$ onto the tangent space of V^r. It can be checked that the operator ∇^V satisfies the properties (47) and is said to be an induced connection. Let X, Y and Z be tangent to V^r, hence one has $\mathtt{Q}\{X\} = 0$, etc. and further $\mathtt{Q}\{[X, Y]\} = 0$. This is shown as follows. Extending the vectors X and Y to the vectors in M^n, which is accomplished just by adding 0 components in the space perpendicular to V^r, we consider $[X, Y]$. By the torsion-free of the Riemannian connection ∇^M, one has $\mathtt{Q}\{[X, Y]\} = \mathtt{Q}\{\nabla^M_X Y - \nabla^M_Y X\} = 0$, which is verified by using expression (48). In fact, all the terms including the terms Γ^k_{ij} cancel out with the symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ and the remaining terms are within the space V^r. Hence, $\nabla^M_X Y - \nabla^M_Y X = \mathtt{P}\{\nabla^M_X Y - \nabla^M_Y X\} = \nabla^V_X Y - \nabla^V_Y X$. Thus, it is found that the connection ∇^V is also torsion-free:

$$\nabla^V_X Y - \nabla^V_Y X = [X, Y]. \tag{67}$$

Therefore the connection ∇^V is also Riemannian. The first condition (55) is satisfied by (67). The second condition (56) is also valid for ∇^V.

Now, *the second fundamental form* $S(X,Y)$ is defined by

$$\nabla^M_X Y = \nabla^V_X Y + S(X,Y), \quad X,Y \in TV^r, \tag{68}$$

which is called the *Gauss' formula*. It is not difficult to see that the function $S(X,Y)$ satisfies the following relation, which is found to be symmetric with respect to X and Y,

$$S(X,Y)(:= \nabla^M_X Y - \nabla^V_X Y) = \mathtt{Q}\{\nabla^M_X Y\} = \mathtt{Q}\{\nabla^M_Y X\} = S(Y,X). \tag{69}$$

This is a Riemannian generalization of Gauss's surface equation. Corresponding to ∇^M and ∇^V, we have two kinds of curvature tensors, $R^M(X,Y)Z$ and $R^V(X,Y)Z$, respectively. Using definition (60) of $R(X,Y)Z$ and the above relations (68) and (69), one can show the following *Gauss equation*:

$$\begin{aligned}\langle W, R^M(X,Y)Z\rangle = \langle W, R^V(X,Y)Z\rangle &+ \langle S(X,Z), S(Y,W)\rangle \\ &- \langle S(X,W), S(Y,Z)\rangle,\end{aligned} \tag{70}$$

where $X,Y,Z,W \in TV^r$. This can be shown by using the definitive relation (66) repeatedly. For example, we have

$$\nabla^V_X \nabla^V_Y Z = \nabla^M_X(\nabla^M_Y Z - \mathtt{Q}\{\nabla^M_Y Z\}) - \mathtt{Q}\{\nabla^M_X(\nabla^M_Y Z - \mathtt{Q}\{\nabla^M_Y Z\})\}.$$

Taking the scalar product with $W \in TV^r$, we obtain

$$\langle W, \nabla^V_X \nabla^V_Y Z\rangle = \langle W, \nabla^M_X \nabla^M_Y Z\rangle - \langle W, \nabla^M_X S(Y,Z)\rangle \tag{71}$$

$$= \langle W, \nabla^M_X \nabla^M_Y Z\rangle + \langle S(X,W), S(Y,Z)\rangle. \tag{72}$$

The last equality can be shown by using (i) $\langle W, \nabla^M_X S(Y,Z)\rangle + \langle \nabla^M_X W, S(Y,Z)\rangle = X\langle W, S(Y,Z)\rangle = 0$, (ii) $W \perp S(Y,Z)$ and (iii) $\mathtt{Q}\{\nabla^M_X W\} = S(X,W)$. We can derive similar expressions for the other terms. Using those expressions, we can verify (70). For $r = n-1$, some detailed descriptions are seen in Frankel (1997) and Abraham and Marsden (1978).

3.8. Geodesic equation

3.8.1. Local coordinates representation

A curve $\gamma(t)$ on M^n is said to be *geodesic* if its tangent $T = \mathrm{d}\gamma/\mathrm{d}t = \dot{\gamma}$ is displaced in parallel along the curve of the parameter t:

$$\frac{\boldsymbol{\nabla} T}{\mathrm{d}t} = \boldsymbol{\nabla}_T T = \frac{\boldsymbol{\nabla}}{\mathrm{d}t}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) = 0.$$

In local coordinates $\gamma = (x^i)$, we have $T = T^i \partial_i = \mathrm{d}\gamma/\mathrm{d}t = (\mathrm{d}x^i/\mathrm{d}t)\partial_i$, and

$$\frac{\boldsymbol{\nabla} T}{\mathrm{d}t} = \nabla_T T = \left[\frac{\mathrm{d}T^i}{\mathrm{d}t} + \Gamma^i_{jk} T^j T^k\right] \partial_i = 0. \tag{73}$$

Thus we obtain the *geodesic equation*:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}+\Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}=0. \tag{74}$$

Using the definition property $\nabla_T T=0$ of the geodesic curve $\gamma_t=\gamma(t)$ and the invariance of the scalar product of parallel-translated vector fields along γ_t, one can interpret the covariant derivative $\nabla_T X$ at γ_0 as the time derivative of the vector $P_t^{-1}X_t$ which is defined by parallel-translating the vector $X_t := X(\gamma_t)$ back to γ_0 along the curve γ_t. In fact, applying the second condition (56) of the Riemannian connection for the vector $Y=Z=T$ (the tangent vector), we have $T\langle X,T\rangle=\langle\nabla_T X,T\rangle$, since $\langle X,\nabla_T T\rangle=0$. The left-hand side is

$$\begin{aligned}T\langle X,T\rangle|_{\gamma_0}&=\lim_{t\to 0}\frac{1}{t}\,(\langle X_t,T_t\rangle-\langle X_0,T_0\rangle)=\lim_{t\to 0}\frac{1}{t}\,(\langle P_t^{-1}X_t,T_0\rangle-\langle X_0,T_0\rangle)\\&=\left\langle\frac{\mathrm{d}}{\mathrm{d}t}P_t^{-1}X_t|_{\gamma_0},T_0\right\rangle,\quad\text{which is equal to }\langle\nabla_T X,T\rangle|_{\gamma_0}.\end{aligned} \tag{75}$$

3.8.2. Group-theoretic representation

On the Riemannian manifold of invariant metric, another formulation of the geodesic equation is possible, because most dynamical systems considered below are equipped with invariant metrics (with respect to either right or left translation). In such cases, the following derivation would be useful. In terms of the adjoint operator $\mathrm{ad}_X Z=[X,Z]$ introduced in (25), let us define the *coadjoint* operator ad^* by

$$\langle\mathrm{ad}^*_X Y,Z\rangle=\langle Y,\mathrm{ad}_X Z\rangle=\langle Y,[X,Z]\rangle. \tag{76}$$

Then Eq. (58) is transformed to $2\langle\nabla_X Y,Z\rangle=\langle\mathrm{ad}_X Y,Z\rangle-\langle\mathrm{ad}^*_Y X,Z\rangle-\langle\mathrm{ad}^*_X Y,Z\rangle$. The non-degeneracy of the inner product given in (45) leads to

$$\nabla_X Y=\frac{1}{2}(\mathrm{ad}_X Y-\mathrm{ad}^*_X Y-\mathrm{ad}^*_Y X). \tag{77}$$

Thus, by taking the condition of parallel displacement of the tangent vector X, an another form of the geodesic equation is given by $\nabla_X X=-\mathrm{ad}^*_X X=0$, since $\mathrm{ad}_X X=[X,X]=0$. In particular, the geodesic equation of a time-dependent problem is represented as

$$\nabla_{\hat{X}}X=\partial_t X+\nabla_X X=\partial_t X-\mathrm{ad}^*_X X=0 \tag{78}$$

for the spatial part X from (54). It should be noted that there is a difference by the sign $\pm$ in the expression of the commutator of the Lie algebra whether the vector fields are left-invariant or right-invariant, as illustrated in Section 2.6 and the footnote given there. When the time evolution of the left-invariant fields such as the rotation group considered in Section 2.6.1 or Section 4 is concerned, the negative sign should be added to both $\mathrm{ad}_X Y$ and $[\cdot,\cdot]$ to derive the time evolution equation. In this regard, it is instructive to see the negative sign in front of the term $(\mathbf{ab}-\mathbf{ba})$ of (28) (and the footnote to definition (25) of ad). This requires that $\nabla^{(\mathrm{L})}_X Y$ should be defined by using the $[\cdot,\cdot]^{(\mathrm{L})}$ defined in (29) in place of $[\cdot,\cdot]$.

In the case of the right-invariant field, the commutator is the Poisson bracket $\{X,Y\}$ of (38) and the time evolution is represented by (78).

3.9. Jacobi equation

Let $C_0:\gamma_0(s)$ be a geodesic curve with the length parameter $s\in[0,L]$, and $C_\alpha:\gamma(s,\alpha)$ a varied geodesic curve where $\alpha\in(-1,+1)$ is a variation parameter and $\gamma_0(s)=\gamma(s,0)$ with s being the arc length for $\alpha=0$. [14] Because $\gamma(s,\alpha)$ is a geodesic, we have $\nabla(\partial_s\gamma)/\partial s=0$ for all α. The function $\gamma(s,\alpha)$ is a differentiable map $\gamma:U\subset\mathbb{R}^2\to M^n$ with the property $[\partial/\partial s,\partial/\partial\alpha]=0$ in U, since $\alpha=\text{const}$ and $s=\text{const}$ are considered as coordinate curves (see (40)). In this circumstance, the following two identities are known:

$$\left(\frac{\nabla}{\partial s}\right)\partial_\alpha\gamma=\left(\frac{\nabla}{\partial\alpha}\right)\partial_s\gamma, \tag{79}$$

$$\frac{\nabla}{\partial s}\left(\frac{\nabla Z}{\partial\alpha}\right)-\frac{\nabla}{\partial\alpha}\left(\frac{\nabla Z}{\partial s}\right)=R(\partial_s\gamma,\partial_\alpha\gamma)Z, \tag{80}$$

(see Frankel, 1997), where $\partial_s\gamma=\partial\gamma/\partial s$ and $\partial_\alpha\gamma=\partial\gamma/\partial\alpha$. Along the reference geodesic $\gamma_0(s)$, let us use the notation $T=\partial_s\gamma$ for the tangent to the geodesic and $J=\partial_\alpha\gamma$ $(\alpha=0)$ for the variation vector. Using $\nabla T/\partial s=0$ and the above identities with $Z=T$, we have

$$\begin{aligned}0&=\frac{\nabla}{\partial\alpha}\frac{\nabla T}{\partial s}=\frac{\nabla}{\partial s}\frac{\nabla T}{\partial\alpha}-R(T,J)T=\frac{\nabla}{\partial s}\frac{\nabla}{\partial\alpha}\partial_s\gamma+R(J,T)T\\&=\frac{\nabla}{\partial s}\frac{\nabla}{\partial s}\partial_\alpha\gamma+R(J,T)T=\frac{\nabla}{\partial s}\frac{\nabla}{\partial s}J+R(J,T)T.\end{aligned}$$

Thus we have obtained the *Jacobi equation* for the geodesic variation J,

$$\frac{\nabla}{\partial s}\frac{\nabla}{\partial s}J+R(J,T)T=0. \tag{81}$$

The variation vector field J is called *Jacobi field*. Defining $\|J\|^2:=\langle J,J\rangle$ and differentiating it two times with respect to s and using (81) and (56), we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{\|J\|^2}{2}=\|\nabla_T J\|^2-K(T,J), \tag{82}$$

where $\nabla_T J=\nabla J/\partial s$, and

$$K(T,J):=\langle J,R(J,T)T\rangle=R_{ijkl}J^iT^jJ^kT^l \tag{83}$$

is a sectional curvature factor associated with the two-dimensional section spanned by J and T. The *sectional curvature* is usually defined by

$$\frac{K(T,J)}{\|T\|^2\|J\|^2-\langle T,J\rangle^2},$$

which reduces to $K(T,J)$ when T and J are orthonormal. Writing $J=\|J\|\boldsymbol{e}_J$ where $\|\boldsymbol{e}_J\|=1$, Eq. (82) is transformed to

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\|J\|=(\|\nabla_T\boldsymbol{e}_J\|^2-K(T,\boldsymbol{e}_J))\|J\|. \tag{84}$$

[14] In this section, the variable s is used in the sense of *length parameter* instead of t.

In the *time-dependent* problem, we use t instead of s and we have

$$\frac{\boldsymbol{\nabla} J}{\partial t} = \partial_t J + \nabla_T J \tag{85}$$

for the spatial pars T and J. However, the curvature tensor $R(J,T)T$ is unchanged and does not include any t-derivative, because the curvature tensor R^{α}_{kij} of (61) vanishes when one of k,i,j takes 0, by definition (62) using (52) with the symmetry $\Gamma^{i}_{0k} = \Gamma^{i}_{k0} = 0$ and the reasonable assumption that the metric tensor and the Christoffel symbols do not depend on t. Thus, the Jacobi equation (81) is replaced by

$$\partial_t^2 J + \partial_t(\nabla_T J) + \nabla_T \partial_t J + \nabla_T \nabla_T J + R(J,T)T = 0. \tag{86}$$

Eq. (86) provides the link between the stability of geodesic curves and the Riemannian curvature, and one of the bases for the geometrical description of dynamical systems considered below.

4. Free rotation of a rigid body

We now consider a physical problem, that is, an application of the geometrical theory formulated in the previous two sections to one of the simplest dynamical systems: Euler's top, i.e. free rotation of a rigid body without external torque. The basic idea is that the governing equation is the geodesic equation over the manifold of a group of transformations SO(3) (a Lie group), which describes the motion of the physical problem. We begin with this simplest system in order to illustrate the underlying geometrical ideas. This section is based on Arnold (1978), Kambe (1998), and Suzuki et al. (1998).

In the mechanics of rigid bodies, *free* rotation is described by *Euler's equations*,

$$J_1 \frac{\mathrm{d}\Omega^1}{\mathrm{d}t} - (J_2 - J_3)\Omega^2\Omega^3 = 0, \quad J_2 \frac{\mathrm{d}\Omega^2}{\mathrm{d}t} - (J_3 - J_1)\Omega^3\Omega^1 = 0$$

$$J_3 \frac{\mathrm{d}\Omega^3}{\mathrm{d}t} - (J_1 - J_2)\Omega^1\Omega^2 = 0, \tag{87}$$

in the body frame (i.e. in the moving frame of reference), where $\Omega = (\Omega^1, \Omega^2, \Omega^3)$ is the angular velocity vector in the frame of the principal axes (x^1, x^2, x^3) of the body's *moment of inertia* $J = (J_{\alpha\beta})$, where J is represented by a diagonal matrix with J_α being its diagonal elements ($\alpha = 1,2,3$). [15]

The angular momentum is given by $M = (M_\alpha) = J\Omega = (J_{\alpha\beta}\Omega^\beta)$, and the kinetic energy K is expressed as

$$K = \frac{1}{2}(M, \Omega)_{R^3} = \frac{1}{2} M_\alpha \Omega^\alpha = \frac{1}{2}(J\Omega, \Omega)_{R^3}, \tag{88}$$

which is invariant during the motion, where $(A,B)_{R^3} = A_\alpha B^\alpha = A_1 B^1 + A_2 B^2 + A_3 B^3$ is the scalar product in $\mathbb{R}^3$. Stability is considered by deriving the Jacobi equation for the geodesic variation vector J.

[15] The moment of inertia is defined by $J_{\alpha\beta} = \int x^\alpha x^\beta \rho \,\mathrm{d}^3\boldsymbol{x}$, where ρ is the mass density assumed constant.

4.1. *Rotation as elements of* SO(3)

Rotation of a rigid body is regarded as a smooth sequence of transformations of the body (e.g. body's principal axes), which are elements of the group of *special orthogonal* transformation of dimension 3: SO(3). By a transformation matrix $A \in \mathrm{SO}(3)$, a point $\mathbf{x}$ (fixed to the body) is moved (mapped) to $\mathbf{x}' = A\mathbf{x}$ with $\det A = 1$. The group $G = \mathrm{SO}(3)$ is a Lie group and consists of all orientation-preserving rotations (i.e. $\det A = +1$) of a rigid body.

An element g of the group G corresponds to a position of the body with its motion arriving at g from the initial position e (the identity). A motion of the body is described by a curve $C: t \to g_t$ on the manifold SO(3) with t the time parameter ($g_0 = e$) (Arnold, 1978, 1966): $g_t = A(t)$, $A(t) \in \mathrm{SO}(3)$, $t \in \mathbb{R}$. An infinitesimal translation $g_{t+\delta t} = g_{\delta t} g_t$ for an infinitesimal time increment δt at g_t is interpreted as a transformation $\delta t \bar{\Omega}$ at $A(t)$, defined by $\delta t \bar{\Omega} \cdot A(t) = A(t+\delta t) - A(t) = A(\delta t)A(t) - A(t) = \delta t \bar{\Omega} \cdot g_t$, where $\bar{\Omega}$ is shown to be an skew-symmetric matrix.

The time derivative $\dot{g}_t = \mathrm{d}A/\mathrm{d}t$ is related to the angular velocity $\bar{\Omega}$ of the body in the (fixed) inertial space and given by $\dot{g}_t = \bar{\Omega} g_t$. On the other hand, the components of the angular velocity Ω in the *moving* body frame (at the identity) are obtained by the left translation of $\dot{g}_t$ by $(g_t^{-1})_* \dot{g}_t = (g_t^{-1})_* \bar{\Omega} g_t$. Using Ω_e, the left-invariant representation is given by $\Omega^{(\mathrm{L})} = (g_t)_* \Omega_e$. The axial vectors equivalent to Ω (or $\bar{\Omega}$) are denoted as $\hat{\Omega}$ (or $\hat{\bar{\Omega}}$). The Ω (or $\hat{\Omega}$) is a *tangent* vector at the identity e of the group G and an element of the Lie algebra $\mathbf{so}(3)$. The space of such vectors is denoted by $T_e G = \mathbf{so}(3)$. According to (29) and (27), the appropriate commutator $[\cdot,\cdot]^{(\mathrm{L})}$ of this algebra has a representation of the vector product in $\mathbb{R}^3$:

$$[X, Y]^{(\mathrm{L})} := \hat{X} \times \hat{Y} \quad \text{for } X, Y \in T_e G = \mathbf{so}(3). \tag{89}$$

The kinetic energy of the body motion is given by the scalar product of the angular velocity vector $\hat{\bar{\Omega}}$ and the angular momentum $\bar{J}\hat{\bar{\Omega}}$ multiplied by $\frac{1}{2}$, where $\bar{J}$ is the moment of inertia in the fixed space. The kinetic energy is a scalar of frame-independent. In other words, it does not depend on the coordinate transformation of left-translation mentioned above. Hence the energy E gives a left-invariant Riemannian metric on the group: $E = \frac{1}{2}(\bar{J}\hat{\bar{\Omega}}, \hat{\bar{\Omega}})_{R^3} = \frac{1}{2}(J\hat{\Omega}, \hat{\Omega})_{R^3}$, where $J = (J_{ij}) = (g_t)^{-1}\bar{J} g_t$ is a diagonal matrix (in the principal axes) with positive elements $J_\alpha (> 0)$. The angular momentum $M = J\Omega$ (in the body system) is a covector. Now one can define the metric $\langle \cdot, \cdot \rangle$ on $T_e G$ by

$$\langle X, Y \rangle := (J\hat{X}, \hat{Y})_{R^3} \quad \text{for } X, Y \in T_e G. \tag{90}$$

Then the kinetic energy is given by $E = \frac{1}{2}\langle \Omega, \Omega \rangle$. The group G is a Riemannian manifold endowed with the left-invariant metric (90).

4.2. *Geometry of the rigid body motion*

Let us consider the geodesic equation on the manifold SO(3). We have already introduced the commutator (89) and the inner product (90) together with the moment of inertia J in diagonal form. Further, the metric is left-invariant, that is, the metric (90) is conserved by the left-translation on the Lie group SO(3). In such a case of invariant metric, the connection satisfies Eq. (58). In terms of the operators ad and ad*, we have expression (77) for $\nabla_X Y$ with $X, Y \in \mathbf{so}(3)$. By using the definitions of $[X, Y]^{(L)} = \hat{X} \times \hat{Y}$ and of $\langle \mathrm{ad}^*_X Y, Z\rangle = \langle Y, [X, Z]^{(L)}\rangle$, the $\mathrm{ad}^*_X Y$ satisfies

$$\langle \mathrm{ad}^*_X Y, Z\rangle = (J\hat{Y}, \hat{X} \times \hat{Z})_{R^3} = (J\hat{Y} \times \hat{X}, \hat{Z})_{R^3} = \langle J^{-1}(J\hat{Y} \times \hat{X}), Z\rangle. \tag{91}$$

Hence, the non-degeneracy of the metric leads to $\mathrm{ad}^*_X Y = J^{-1}(J\hat{Y} \times \hat{X})$. Thus it is found from (77) that, (Kambe, 1998; Suzuki et al., 1998),

$$\nabla_X Y = \frac{1}{2} J^{-1}(J(\hat{X} \times \hat{Y}) - (J\hat{X}) \times \hat{Y} - (J\hat{Y}) \times \hat{X}) = \frac{1}{2} J^{-1}(\tilde{K}\hat{X} \times \hat{Y}), \tag{92}$$

where $\tilde{K}$ is a diagonal matrix with the diagonal elements, $\tilde{K}_\alpha := -J_\alpha + J_\beta + J_\gamma$ for $(\alpha, \beta, \gamma) = (1, 2, 3)$ or its cyclic permutation (all $\tilde{K}_\alpha > 0$ owing to the fundamental definition of the moment of inertia).

The tangent vector at the identity e is the angular velocity $\hat{\Omega}$. The geodesic equation of the time-dependent problem is given by (78) with $X = \hat{\Omega}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Omega} - J^{-1}((J\hat{\Omega}) \times \hat{\Omega}) = 0, \tag{93}$$

which is also written as $(\mathrm{d}/\mathrm{d}t)J\hat{\Omega} = (J\hat{\Omega}) \times \hat{\Omega}$. This is nothing but the Euler equation (87).

Equation of the *Jacobi* field $\hat{Y}$ along the geodesic generated by $\hat{X}$ is given by (86) with the spatial parts $(\hat{X}, \hat{Y})$ in place of (T, J). (Here the hat symbol denotes the spatial axial vector.) Expression (92) is rewritten as

$$\nabla_X Y = (1/2)[(\hat{X} \times \hat{Y}) - F(\hat{X}, \hat{Y})],$$

where $F(\hat{X}, \hat{Y}) = J^{-1}(J\hat{X} \times \hat{Y} + J\hat{Y} \times \hat{X}) = F(\hat{Y}, \hat{X})$. Then, the Jacobi equation (86) reduces to

$$\frac{\mathrm{d}^2\hat{Y}}{\mathrm{d}t^2} + \hat{X} \times \frac{\mathrm{d}\hat{Y}}{\mathrm{d}t} - F\left(\hat{X}, \frac{\mathrm{d}\hat{Y}}{\mathrm{d}t}\right) + \frac{1}{2}F(\hat{X}, \hat{X}) \times \hat{Y} - F(\hat{X}, \hat{X} \times \hat{Y}) = 0.$$

To calculate the curvature tensor, we apply the expression of the connection ∇ of (92) repeatedly in formula (60) together with the definition of the commutator (89). Finally it is found that

$$R(X, Y)Z = -\frac{1}{4J_1J_2J_3}(\tilde{M}(\hat{X} \times \hat{Y})) \times J\hat{Z}, \tag{94}$$

where $\tilde{M} = \mathrm{diag}(\tilde{M}_1, \tilde{M}_2, \tilde{M}_3)$ is a diagonal matrix of third order with the diagonal element $\tilde{M}_\alpha = -3J_\alpha^2 + (J_\beta - J_\gamma)^2 + 2J_\alpha(J_\beta + J_\gamma)$ with $(\alpha, \beta, \gamma) = (1, 2, 3)$ and its cyclic permutation.

Defining the unit basis vectors as $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ in the principal-axes system and writing the sectional curvature $K(\mathbf{e}_i, \mathbf{e}_j)$ (see (83)) as K_{ij} in short, one obtains $K_{12} = \tilde{M}_3/4J_3$, $K_{23} = \tilde{M}_1/4J_1$, $K_{31} = \tilde{M}_2/4J_2$.

For the stable steady rotation (with respect to the axis of the largest or least value of J_α), it is found that the curvatures K_{ij} take either positive values always, or both positive and negative values in oscillatory manner, depending on the tensor J. However, it is found that the time average $\bar{K}$ is always positive for any J in the linearly stable case, while there exist J's which make $\bar{K}$ negative in the case of linear instability (Suzuki et al., 1998). The results are consistent with the known properties of rotating rigid bodies in *mechanics*.

5. Geodesic equation on $D(S^1)$ and KdV equation on $\hat{D}(S^1)$

Now we consider a second example of application of the geometrical theory. This is a dynamical system of smooth mapping along a circle and found to be a fundamental problem in physics. The geodesic is a curve on the manifold of orientation-preserving diffeomorphisms of a circle S^1 (with a periodicity T) noted in Section 2.7. Two systems are considered below: the first one is the geodesic equation over a group of diffeomorphisms $D(S^1)$, describing a simple diffeomorphic flow on S^1, and the second one is the KdV equation, which is the geodesic equation over an extended group $\hat{D}(S^1)$. The manifold S^1 is spatially one-dimensional, but its diffeomorphism has infinite degrees of freedom because the pointwise mapping describes arbitrary (but orientation-preserving) deformation of the circle.[16]

We consider infinite-dimensional Lie groups $D(S^1)$ and $\hat{D}(S^1)$, including an infinite-dimensional algebra called the Virasoro algebra (de Azcárraga and Izquierdo, 1995). This part is based on Ovsienko and Khesin (1987), Misiołek (1997), and Kambe (1998).

5.1. *Simple diffeomorphic flow (dispersionless KdV)*

Consider a group $D(S^1)$ of diffeomorphisms of a circle S^1 of periodicity T, equipped with a *right-invariant* metric (see Sections 2.7 and 3.2.2). Because of this metric invariance, the Riemannian connection ∇ is given by expression (77): $\nabla_X Y = \frac{1}{2}(\mathrm{ad}_X Y - \mathrm{ad}^*_Y X - \mathrm{ad}^*_X Y)$, where X and Y are two vectors in the tangent space at id, i.e. $X, Y \in T_{\mathrm{id}}D(S^1)$. The vectors X and Y are also considered to be two tangent vector fields of TS^1 (the tangent bundle of S^1).

Using the tangent fields $X = u(x)\partial_x,\ Y = v(x)\partial_x,\ Z = w(x)\partial_x \in TS^1$ of the S^1 diffeomorphisms (see Section 2.7) and definition (43) of the (Witt) algebra, we have $\mathrm{ad}_X Y = [X, Y] = (uv' - vu')\partial_x$, where $u' = \partial_x u = u_x$. Then the $\mathrm{ad}^*_X Y$ is given by

$$\langle \mathrm{ad}^*_X Y, Z\rangle = \langle Y, \mathrm{ad}_X Z\rangle = \int_{S^1} v(uw' - wu')\,\mathrm{d}x = -\int_{S^1} (uv' + 2vu')w\,\mathrm{d}x,$$

where the definition of metric (46) is used and integration by parts is performed with the periodic boundary conditions $u(x + T) = u(x)$, etc. Hence, one obtains

$$\mathrm{ad}^*_X Y = -(uv' + 2vu')\partial_x.$$

It is found from (77) that the Riemannian connection on the group manifold $D(S^1)$ is given by

$$\nabla_X Y = (2uv' + vu')\partial_x.$$

The geodesic equation is represented by (78):

$$\partial_t X + \nabla_X X = \partial_t X - \mathrm{ad}^*_X X = 0.$$

Thus it is found that the geodesic equation on the manifold $D(S^1)$ is given by

$$u_t + 3uu_x = 0.$$

[16] The derivation on $D(S^1)$ (or $\hat{D}(S^1)$) in the present section can be generalized to the group $D^s(\mathbb{R}^1)$ (or $\hat{D}(\mathbb{R}^1)$) of diffeomorphisms of Sobolev class $H^s(\mathbb{R})$ (see Section 7 and Appendix A.2). Cauchy problem is known to be well-posed for non-periodic case as well in the Sobolev space $H^s(\mathbb{R})$ for any $s > 3/2$. (See Kato, 1983; Himonas and Misiołek, 2001.)

Compared with the KdV equation of the form, $\partial\bar{u}/\partial t + \bar{u}\partial\bar{u}/\partial x + \kappa\bar{u}_{xxx} = 0$, this equation has no third-order dispersion term $\kappa\bar{u}_{xxx}$, where κ is a constant and $\bar{u} = 3u$. The third-order derivative term is only introduced by the central extension considered in the next section. The above equation would be termed as the one governing a *simple diffeomorphic* flow.

5.2. Central extension

An element g of the diffeomorphism group D represents a map $g : x \in S^1 \to g(x) \in S^1$. We may write $x = e^{i\phi}$ and consider instead the mapping $\phi \mapsto g(\phi)$ such that $g(\phi + 2\pi) = g(\phi) + 2\pi$. Then, we define the transformed function $f_g(\phi) := e^{ig(\phi)}$ with $f_e = e^{i\phi}$. Further, associated with the group D, we may consider a phase shift $\eta(g) : D \to \mathbb{R}$ in a new transformed function F_g, such that the mapping $\phi' = g(\phi)$ transforms $F_g(\phi')$ into $\exp[i\eta(g)]\exp[ig(\phi)]$. This results in the central extension described in the Appendix A.3.

An extension of the group D is denoted by $\hat{D}$, and its elements are written as

$$\hat{f} := (f, a), \quad \hat{g} := (g, b) \in \hat{D}(S^1)$$

for $f, g \in D(S^1)$ and $a, b \in \mathbb{R}$, where $\hat{D}(S^1) = D(S^1) \oplus \mathbb{R}$. The group operation is represented by (see (A.3.10))

$$\hat{g} \circ \hat{f} := (g \circ f, a + b + B(g, f)), \tag{95}$$

$$B(g, f) := \frac{1}{2}\int_{S^1} \ln \partial_x(g \circ f) \mathrm{d} \ln \partial_x f, \tag{96}$$

where $B(g, f)$ is the Bott cocycle (Appendix A.3). It can be readily shown that the following subgroup $\hat{D}_0$ is a *center* of the extended group $\hat{D}$, where $\hat{D}_0$ is defined by $\{\hat{f}_0 \mid \hat{f}_0 = (\mathrm{id}, a),\ a \in \mathbb{R}\}$, and $\mathrm{id}(x) = x$.

5.3. KdV equation as a geodesic equation on $\hat{D}(S^1)$

We now consider the geodesic equation on the extended manifold $\hat{D}(S^1)$ (Ovsienko and Khesin, 1987). Let $t \mapsto \hat{\xi}_t := (\xi_t, \alpha t)$ be a flow starting at $\hat{\xi}_0 = (\mathrm{id}, 0) =: \widehat{\mathrm{id}}$ in the direction $\dot{\hat{\xi}}_0 = (u(x)\partial_x, \alpha)$, and the second flow $t \mapsto \hat{\eta}_t = (\eta_t, \beta t)$ starting at $\widehat{\mathrm{id}}$ in the direction $\dot{\hat{\eta}}_0 = (v(x)\partial_x, \beta)$, where $\alpha, \beta \in \mathbb{R}$. Non-trivial central extension of $T_{\mathrm{id}}D(S^1)$ to $T_{\widehat{\mathrm{id}}}\hat{D}(S^1)$ is the *Virasoro algebra* (de Azcárraga and Izquierdo, 1995), in which the extended component is the general 2-cocycle $c(u, v)$ found by Gelfand and Fuchs (1968). For any two tangent fields

$$\hat{u} = (u(x,t)\partial_x, \alpha), \quad \hat{v} = (v(x,t)\partial_x, \beta) \in T_{\widehat{\mathrm{id}}}\hat{D}(S^1),$$

the *commutator* is given by

$$[\hat{u}, \hat{v}] := ((u\partial_x v - v\partial_x u)\partial_x, c(u, v)), \tag{97}$$

$$c(u, v) := \int \partial_x^2 u \partial_x v \, \mathrm{d}x = -c(v, u), \tag{98}$$

where $c(u,v)$ is the Gelfand–Fuchs cocycle. Derivation of $c(u,v)$ from $\alpha t+\beta t+B(\eta_t,\xi_t)$ is shown in Kambe (2001). The *metric* is defined by

$$\langle\hat{u},\hat{v}\rangle := \int u(x)v(x)\,\mathrm{d}x+\alpha\beta. \tag{99}$$

Following the procedure of Section 5.1, the covariant derivative is derived as

$$\nabla_{\hat{u}}\hat{v}=\left(w\partial_x,\frac{1}{2}\int_{S^1}u_{xx}v_x\,\mathrm{d}x\right),$$

$$w=2uv_x+vu_x+\frac{1}{2}(\alpha v_{xxx}+\beta u_{xxx}).$$

The geodesic equation is written as $\partial\hat{u}/\partial t+\nabla_{\hat{u}}\hat{u}=0$, which leads to the following two equations:

$$u_t+3uu_x+\alpha u_{xxx}=0,$$

$$\partial_t\alpha=0. \tag{100}$$

The second equation follows from the property, $\int_{S^1}u_{xx}u_x\,\mathrm{d}x=0$. Thus, the *KdV equation* is derived, where the constant α is called the *central charge*. It is to be noted that the central extension is described by a phase-shift $\eta(g)$ associated with the transformation g, and the charge α is associated with the rate of phase-shift, i.e. a translational motion. This reminds us that the KdV equation is derived originally for the motion of a long wave in a shallow water, where the fluid particles move translationally.

5.4. Sectional curvatures of KdV system

The geometrical theory leads to a relationship between the stability of geodesic curves on a Riemannian manifold and its curvature. The link is expressed by the Jacobi equation for geodesic variation J in Section 3.9. An evolution equation for its norm $\|J\|$ is given by Eq. (82), where the second term on the right-hand side $K(J,T)$ is the sectional curvature associated with the two-dimensional section spanned by J and T. If $K(J,T)$ is negative, the right-hand side is positive. Then exponential growth of the magnitude $\|J\|$ is predicted, which is understood that the geodesics are *unstable*.

In this context, the sectional curvature of the KdV system is estimated (Misiołek, 1997; Kambe, 1998) for the section spanned by the two tangent vectors (with a common central charge α), $\hat{u}=(u(x,t)\partial_x,\alpha)$ and $\hat{v}=(v(x,t)\partial_x,\alpha)$:

$$K(\hat{u},\hat{v})=\frac{1}{4}F-9G-\frac{3}{4}H^2,$$

where

$$F=\int_{S^1}(\alpha(u'''-v''')+2(vu'-uv'))^2\,\mathrm{d}x,$$

$$G=\alpha\int_{S^1}u'v'(u''+v'')\,\mathrm{d}x,\quad H=\int_{S^1}u''v'\,\mathrm{d}x.$$

For the sinusoidal fields $\hat{u}_n = (a_n \sin nx\partial_x, \alpha)$ and $\hat{v}_n = (b_n \cos nx\partial_x, \alpha)$ with $T = 2\pi$, it is found that

$$K(\hat{v}_1, \hat{u}_n) = \frac{\pi}{4}(\alpha^2 b_1^2 + \alpha^2 (a_n n^3)^2 + 2(b_1 a_n)^2 (1 + n^2)) > 0 \quad \text{for } n \geqslant 3,$$

$$K(\hat{v}_1, \hat{v}_n) = \frac{\pi}{4}(\alpha^2 b_1^2 + \alpha^2 (b_n n^3)^2 + 2(b_1 b_n)^2 (1 + n^2)) > 0 \quad \text{for } n \geqslant 3.$$

Therefore, both of the sectional curvatures $K(\hat{v}_1, \hat{u}_n)$ and $K(\hat{v}_1, \hat{v}_n)$ are *positive* for $n \geqslant 3$. Thus, most of the sectional curvatures are positive. However, there are some sections which are not always positive. In fact,

$$K(\hat{v}_1, \hat{u}_1) = \frac{\pi}{4}(a_1 b_1)^2 \left(-3\pi + 8 + \alpha^2 \frac{a_1^2 + b_1^2}{a_1^2 b_1^2} \right).$$

Similarly it can be shown that $K(\hat{v}_n, \hat{u}_n)$ is not always positive for any integer n as well.

6. Geometrical theory of chaos in a Hamiltonian system

A self-gravitating system of N point masses is one of the typical dynamical systems studied in the conventional analytical dynamics. As a third example, the differential geometric formulation is given to this system of finite degrees of freedom (Pettini, 1993). A simplest non-trivial case is the Hénon–Heiles system, a two-degrees-of-freedom Hamiltonian system, which is well known to be a chaotic system. Within the present framework, stability of the trajectories of the dynamical system is studied when the Riemannian curvatures of the manifold are known. This leads to the geometric characterization of *Hamiltonian chaos* (Cerruti-Sola and Pettini, 1996).

It has recently been revealed that the phenomenon of phase transitions is related at a deep level to a change in the topology of configuration space of the system. Fluctuations of the configuration-space curvature exhibit a singular behavior at the phase transition. In this section, only the former chaos analysis is presented. As to the latter subject of geometrical theory of phase transition, only the following reference is given here: Casetti et al. (2000).

6.1. A dynamical system with self-gravitation

Consider a dynamical system described by the Lagrangian function,

$$L(\bar{q}, \dot{\bar{q}}) := E - V = \frac{1}{2} a_{ij}(\bar{q}) \dot{q}^i \dot{q}^j - V(\bar{q}), \tag{101}$$

where $\bar{q} := (q^1, \ldots, q^N)$ and $\dot{\bar{q}} := (\dot{q}^1, \ldots, \dot{q}^N)$ are the coordinates and velocities of the N degrees of freedom system, and $V(\bar{q})$ is the potential of self-gravitation. The first term $E=(1/2)a_{ij}\dot{q}^i\dot{q}^j$ represents the kinetic energy and a_{ij} $(i, j = 1, \ldots, N)$ is the mass tensor. We consider only the case of $a_{ij} = \delta_{ij}$ (Kronecker's delta). The Hamiltonian H is represented by $H := p_\alpha \dot{q}^\alpha - L(\bar{q}, \dot{\bar{q}}) = (1/2) a^{\alpha\beta} p_\alpha p_\beta + V(\bar{q}) = E + V$ where $p_\alpha = a_{\alpha i} \dot{q}^i$, and $(a^{\alpha\beta})$ is the inverse of $(a_{ij}) = \underline{\underline{a}}$, i.e. $(a^{\alpha\beta}) = \underline{\underline{a}}^{-1}$.

One can introduce an enlarged Riemannian manifold, in terms of the Eisenhart metric g^E $(M^N \times \mathbb{R}^2)$ by introducing two additional coordinates q^0 $(=t)$ and q^{N+1} (Eisenhart, 1929; Pettini, 1993). Defining

$Q := (q^0, \bar{q}, q^{N+1})$, the arc length $\mathrm{d}s$ is represented by

$$\mathrm{d}s^2 = g^E_{ij}(Q)\,\mathrm{d}Q^i\,\mathrm{d}Q^j = a_{ij}\,\mathrm{d}q^i\,\mathrm{d}q^j - 2V(\bar{q})\,\mathrm{d}q^0\,\mathrm{d}q^0 + 2\mathrm{d}q^0\,\mathrm{d}q^{N+1},$$

$(q^0 = t)$, and the metric tensor $g^E = g^E_{ij}(Q)$ is represented by, for $i, j = 0, \ldots, N+1$,

$$g^E := \begin{pmatrix} -2V(\bar{q}) & \underline{0} & 1 \\ \underline{0}^{\mathrm{T}} & \underline{\underline{a}} & \underline{0}^{\mathrm{T}} \\ 1 & \underline{0} & 0 \end{pmatrix}, \quad (g^E)^{-1} = \begin{pmatrix} 0 & \underline{0} & 1 \\ \underline{0}^{\mathrm{T}} & \underline{\underline{a}}^{-1} & \underline{0}^{\mathrm{T}} \\ 1 & \underline{0} & 2V(\bar{q}) \end{pmatrix}, \tag{102}$$

where $\underline{\underline{a}} := (a_{ij})$, $\underline{0}$ is the null row vector and $\underline{0}^{\mathrm{T}}$ is its transpose.

The Christoffel symbols Γ^k_{ij} are given by (65) with the metric tensors g^E. Since the matrix elements $a_{ij}(=\delta_{ij})$ are constant, only non-vanishing Γ^k_{ij} are

$$\Gamma^i_{00} = g^{il}\frac{\partial V}{\partial q^l} = \partial_i V, \quad \Gamma^{N+1}_{0i} = \Gamma^{N+1}_{i0} = -g^{0N+1}\frac{\partial V}{\partial q^i} = -\partial_i V \tag{103}$$

$(i = 1, \ldots, N)$. The natural motions are obtained as the projection on the space–time configuration space $(t, \bar{q})$ and given by the geodesics satisfying $\mathrm{d}s^2 = k^2\,\mathrm{d}t^2$ and $\mathrm{d}q^{N+1} = (k^2/2 - L(\bar{q}, \dot{\bar{q}}))\,\mathrm{d}t$. The constant k can be always set as $k = 1$, leading to $\mathrm{d}s^2 = \mathrm{d}t^2$ (Casetti et al., 2000).

The geodesic equation is given by the form (74) in local coordinates. Using (103) and $\mathrm{d}s = \mathrm{d}t$ (and $\underline{\underline{a}} = \underline{\underline{a}}^{-1} = (\delta_{ij})$), we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}q^i = -\frac{\partial V}{\partial q^i} \quad (i = 1, \ldots, N), \quad \frac{\mathrm{d}q^0}{\mathrm{d}t} = 1,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}q^{N+1} = 2\frac{\partial V}{\partial q^i}\frac{\mathrm{d}q^i}{\mathrm{d}t} = -\frac{\mathrm{d}L}{\mathrm{d}t} \quad (\text{since } \mathrm{d}T/\mathrm{d}t = -\mathrm{d}V/\mathrm{d}t). \tag{104}$$

Choosing arbitrary constants appropriately, we have $q^0 = t$ and $\mathrm{d}q^{N+1}/\mathrm{d}t = 1/2 - L$. Eq. (104) is the Newton's equation of motion. Thus it is found that the geometric machinery works for the present dynamical system too. The Eisenhart metric (Newtonian limit metric of the general relativity) is chosen here because it is seen just below to have very simple curvature properties, although there is another metric known as the Jacobi metric (Section 3.2.1).

The link between the stability of trajectories and the geometrical characterization of the manifold $(M(\bar{q}) \otimes \mathbb{R}^2, g^E)$ is expressed by the Jacobi equation (81) (rewritten):

$$\left(\frac{\nabla}{\mathrm{d}s}\right)^2 J + R(J, \dot{Q})\dot{Q} = 0. \tag{105}$$

From (62), the non-vanishing components of the curvature tensors are

$$R^i_{0j0} = -R^i_{00j} = \partial_i\partial_j V \quad \text{for } i, j = 1, \ldots, N. \tag{106}$$

The Ricci tensor, defined by $R_{kj} \equiv R^l_{klj}$, has only non-zero component $R_{00} = R^l_{0l0} = \Delta V$. The scalar curvature, defined by $R \equiv g^{ij}R_{ij} = g^{00}R_{00}$, vanishes identically since $g^{00} = 0$ by (102).

It is interesting to find that the Jacobi equation (81) is equivalent to the equation of tangent dynamics, that is the evolution equation of infinitesimal variation vector $\xi(t)$ along the reference trajectory $\bar{q}_0(t)$. In fact, writing the perturbed trajectory as $q^i(t) = q^i_0(t) + \xi^i(t)$ and substituting it to the equation of motion, $\mathrm{d}^2q^i/\mathrm{d}t^2 = -\partial V/\partial q^i$, the linearized perturbation equation with respect to $\xi(t)$ reads

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\xi^i = -\left(\frac{\partial^2 V(\bar{q})}{\partial q^i \partial q^j}\right)_{\bar{q}=\bar{q}_0(t)} \xi^j.$$

This is equivalent to the Jacobi equation (81) by using (59), (103) and (106) because, noting $J^0=0$, one has $(\nabla J/\mathrm{d}s)^i = \mathrm{d}J^i/\mathrm{d}t + \Gamma^i_{00}J^0\dot{Q}^0 = \mathrm{d}J^i/\mathrm{d}t$ and $R(J,\dot{Q})\dot{Q} = (\partial_i\partial_j V)J^j$.

6.2. System of two degrees of freedom (Hénon–Heiles model)

In regard to the information of dynamical behaviors, either regular or chaos, all is included in this geometrical characterization. In order to see this, the previous formulation is applied to a two-degrees-of-freedom system described by the Lagrangian, $L=\frac{1}{2}((\dot{q}_1)^2+(\dot{q}_2)^2)-V(q_1,q_2)$, where we are using the lower suffices such as q_1 (or J_2) in this section only in order to make a concise definition of its square $(q_1)^2=q_1^2$. The enlarged coordinate and velocity are $Q=(t,q_1,q_2,q_3)$ and $\dot{Q}=(1,\dot{q}_1,\dot{q}_2,\dot{q}_3)$, and the geodesic separation vector is $J=(J_0=0,J_1,J_2,J_3)$. On the constant energy surface and along the geodesic $Q(t)$, one can always assume that J is orthogonal to $\dot{Q}$, i.e. $\langle J,\dot{Q}\rangle = g_{ij}J_i\dot{Q}_j=0$. In fact, expressing as $J=J^{\perp}+c\dot{Q}$ and substituting in Eq. (81), it is readily seen that the terms related to the parallel component $c\dot{Q}(=cT)$ drop out and the equation is nothing but the one for the orthogonal component $J^{\perp}$. Further, the components J_3 and $\dot{q}_3$ are irrelevant because $g_{33}=0$.

The equation for the norm of geodesic separation $\|J\|$ is Eq. (82). Cerruti-Sola and Pettini (1996) chose as $J=(0,\dot{q}_2,-\dot{q}_1,0)$. Then the normalized curvature $\hat{K}$ is given by

$$\hat{K}(\dot{Q},Q) := \frac{K(J,\dot{Q})}{\|J\|^2} = \frac{1}{2(E_{\mathrm{total}}-V(\bar{q}))}\left(\frac{\partial^2 V}{\partial q_1^2}\dot{q}_2^2 - 2\frac{\partial^2 V}{\partial q_1\partial q_2}\dot{q}_1\dot{q}_2 + \frac{\partial^2 V}{\partial q_2^2}\dot{q}_1^2\right),$$

($E_{\mathrm{total}}=E+V$, total energy), which is computable on the constant energy surface S_E. They define the integral of negative curvature value $\hat{K}_- = \{\hat{K}\colon \hat{K}(\dot{Q},Q)<0\}$ by

$$\langle \hat{K}_-\rangle := \frac{1}{A(S_E)}\int_{S_E} \hat{K}_-\,\mathrm{d}\bar{q}\,\mathrm{d}\dot{\bar{q}},$$

where $A(S_E)$ is the area of S_E. The quantity $\langle\hat{K}_-\rangle$ was estimated at different energy values E.

In the Hénon and Heiles (1964) model, the Hamiltonian is $H=(1/2)(p_1^2+p_2^2)+V(q_1,q_2)$ and the potential is chosen as $V(q_1,q_2)=\frac{1}{2}(q_1^2+q_2^2)+q_1^2q_2-\frac{1}{3}q_2^3=\frac{1}{2}r^2+\frac{1}{3}r^3\sin 3\theta$, where $q_1=r\cos\theta$ and $q_2=r\sin\theta$. It is shown that the transition from order to chaos is quantitatively described by measuring, on a Poincaré section, the ratio σ of the area covered by the regular trajectories divided by the total area accessible to the motions. For low energies E_{total} the whole area is practically covered by regular orbits and hence the ratio σ is almost 1. As E_{total} is increased, σ decreases very slowly from 1 below $E_{\mathrm{total}}\approx 0.1$. As E_{total} is increased further, σ begins to drop rapidly to very small values. At $E_{\mathrm{total}}=1/6$, the accessible area is marginal because the equi-potential curve $V(q_1,q_2)=1/6$ is an equilateral triangle (including the origin within it). Beyond $E_{\mathrm{total}}=1/6$, the equi-potential curves are open, and the motions are unbounded. Thus the accessible area becomes infinite.

It is shown in Cerruti-Sola and Pettini (1996) that, for low values of $E_{\rm total}$, the integral of the negative curvature $\langle \hat{K}_- \rangle$ is almost zero, but that, at the same $E_{\rm total}$ value (≈ 0.1) at which σ begins to drop rapidly, the value $\langle \hat{K}_- \rangle$ starts to increase rapidly. The exact coincidence between the critical energy level for the σ decreasing below 1 and the one for the $\langle \hat{K}_- \rangle$ increasing above 0 is understood that the onset of sharp increase of chaotic domains is detected by the increase of the negative curvature integral $\langle \hat{K}_- \rangle$. Along with this, the fraction of the area $A_-(S_E)$ where $\hat{K} < 0$ is also estimated as a function of $E_{\rm total}$. The transition is again detected by this quantity too.

7. Flows of an inviscid incompressible fluid

Motion of fluid particles of an inviscid incompressible fluid is described on the basis of the geometrical framework. In the conventional approach, flows of an inviscid fluid are well described already in the fluid dynamics. However, flows of an inviscid fluid are equivalently expressed by the geodesics on the manifold of all volume-preserving diffeomorphisms. This formulation is based on the Riemannian geometry and Lie group theory, developed first by Arnold (1966). The present approach reveals new aspects which are not studied in the conventional fluid dynamics. For example, behaviors of the geodesics are controlled by Riemannian (sectional) curvatures, which are quantitative characterizations of the flow (in infinite many numbers). In particular, the analysis shows that the curvatures are found to be mostly negative (with some exceptions), which can be related to mixing or ergodicity of the fluid motion in a bounded domain.

It is known that the geodesic equation on a central extension of the group of volume preserving diffeomorphisms is equivalent to the flow of a perfectly conducting fluid. Here, only the following references are referred: Vizman (2001) and Zeitlin (1992). The present section is based on the works of Misiołek (1993), Nakamura et al. (1992), Hattori and Kambe (1994), and Ebin and Misiołek (1997).

7.1. Basic concepts

We consider flows of an inviscid incompressible fluid on a manifold M (which is the flow region): $\mathbb{R}^2$ (or T^2), or $\mathbb{R}^3$ (or T^3). The motivation of the geometrical analysis is based on the observation that Euler's equation of motion is a geodesic equation on a group of volume-preserving diffeomorphisms with the metric defined by the kinetic energy. The set of all *volume-preserving* diffeomorphisms of M composes a group manifold $D_\mu(M)$, of which an element $g \in D_\mu(M)$ is a map, $g : M \to M$.

Suppose that a curve $t \to g_t(x)$ denotes a fluid flow, then a point x is mapped to the point $g_t(x)$ at a time t. This is a *Lagrangian* description of flows. Tangent vector field (velocity field) at the time t is represented as

$$U_t(x) := \dot{g}_t(x) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(g_{\Delta t} - \mathrm{id}) \circ g_t(x) = u \circ g_t(x) = u(g_t(x)). \tag{107}$$

By definition, the tangent field $U_t(x) = u \circ g_t(x)$ at g_t is a *right-invariant* field. Right translation with g_t^{-1} yields the velocity field at $\mathrm{id} = g_0$:

$$u_t(x) = \dot{g}_t \circ g_t^{-1}(x), \quad u_t \in T_{\mathrm{id}} D_\mu(M).$$

At the identity $e = \mathrm{id}$ (i.e. at the initial manifold M_0), the velocity field $u_t(x)$ satisfies the divergence-free property, i.e. div $u_t = 0$. The suffix t denotes that the tangent vector u_t is time-dependent. In terms of the fluid-dynamics, $u_t(x)$ represents an *Eulerian* velocity field at a time t, whereas U_{g_t} is its Lagrangian counterpart. In mathematical language, u_t is an element of the Lie algebra $T_{\mathrm{id}}D_\mu(M)$.

It is useful to consider the manifold $D^s_\mu(M)$ which is a subgroup of volume preserving diffeomorphisms (of M) of Sobolev class H^s, where $s > n/2+1$ and $n = \dim M$ (see Appendix A.2). The group $D^s_\mu(M)$ is a weak Riemannian submanifold of the group $D^s(M)$ of all Sobolev H^s-diffeomorphisms of M. An arbitrary vector field $v(x) \in T_e D^s_\mu(M)$ can be decomposed into L^2-orthogonal components of divergence-free part $\bar{v}$ and gradient part (Appendix A.2):

$$v = \bar{v} + \mathrm{grad}\, f, \quad f \in H^{s+1}(M). \tag{108}$$

7.2. Right-invariant field

Before presenting the geometrical theory of hydrodynamics, it is essential to introduce the notion of right-invariant field for its basis. Consider the tangent field $U_\eta \in T_\eta D^s_\mu(M)$ at any $\eta \in D^s_\mu(M)$, and suppose that U_η is *right-invariant*, that is,

$$U_\eta(x) := U_e \circ \eta(x) \quad \text{for } U_e \in T_e D^s_\mu(M).$$

Correspondingly, the right-invariant L^2-metric is defined on $D_\mu(M)$ (not on all of $D(M)$) by

$$\langle U, V\rangle_\eta := \int_M (U_\eta \circ \eta^{-1},\ V_\eta \circ \eta^{-1})_x \, \mathrm{d}\mu[x] = \langle U_e, V_e\rangle_e, \tag{109}$$

where $U_\eta, V_\eta \in T_\eta D^s_\mu(M)$, $\mathrm{d}\mu$ is the volume form and $(\cdot,\cdot)_x$ is the scalar product at each point $x \in M$. Applying the right translation by η on the middle side of (109), we have

$$\begin{aligned}\langle U, V\rangle_\eta &= \int_{\eta(M)} (U_\eta \circ \eta^{-1},\ V_\eta \circ \eta^{-1})_x \circ \eta \, \mathrm{d}\mu \circ \eta \\ &= \int_{x\in M} (U_\eta(x),\ V_\eta(x))_{\eta(x)} \, \mathrm{d}\mu[x].\end{aligned}$$

The second equality is verified by using the volume-preserving property $\mathrm{d}\mu[\eta(x)] = \mathrm{d}\mu[x]$ and by changing the variable from η to x. (Compare with $\mathrm{d}\mu[\eta(x)] = \eta^* \mathrm{d}\mu[x]$ (pull-back) defined in Section 2.4). Thus the present L^2-metric (109) is *isometric* with respect to the right translation by any $\eta \in D^s_\mu(M)$. The metric (109) induces, on $D^s(M)$ and $D^s_\mu(M)$, smooth Riemannian connections $\hat{\nabla}$ and $\bar{\hat{\nabla}} = \mathrm{P}\hat{\nabla}$ respectively, where the symbol P is the projection operator to the divergence-free part. For any right-invariant vector fields $U_\eta, V_\eta \in T_\eta D^s_\mu(M)$, we have the *right-invariant* connection,

$$(\hat{\nabla}_{U_\eta} V_\eta)_\eta := (\nabla_{U_e} V_e)_e \circ \eta \tag{110}$$

for $\eta \in D^s_\mu(M)$, where ∇ is the covariant derivative on M (manifold of Eulerian description). Similarly, we have

$$(\bar{\hat{\nabla}}_{U_\eta} V_\eta)_\eta := \mathrm{P}[\nabla_{U_e} V_e]_e \circ \eta. \tag{111}$$

An arbitrary vector field $X_\eta(x)$ on $D^s(M)$ can be decomposed into L^2-orthogonal components of divergence-free part $\bar{X}_\eta$ and gradient part, by (108) and the isometry of the L^2-metric:

$$X_\eta = \bar{X}_\eta + \text{grad}\, f \tag{112}$$

for any $\eta \in D^s_\mu(M)$, where $\bar{X}_\eta \in T_\eta D^s_\mu(M) = \{X \in T_\eta D^s\colon \ \text{div}(X \circ \eta^{-1}) = 0\}$ and $f \in H^{s+1}(M) \circ \eta$. We denote by $\mathtt{P}_\eta$ and $\mathtt{Q}_\eta$ the orthogonal projections onto the first and the second terms at η in (112) respectively. The difference of the two connections $\hat{\nabla}$ and $\bar{\nabla}$ is the second fundamental form S of $D^s_\mu(M)$:

$$S(U_\eta, V_\eta) := \hat{\nabla}_{U_\eta} V_\eta - \bar{\nabla}_{U_\eta} V_\eta = \mathtt{Q}_\eta[\hat{\nabla}_{U_\eta} V_\eta] \tag{113}$$

(see (69)). This is also right-invariant. For tangent fields $U, V, W, Z \in T_\eta D^s_\mu(M)$, the curvature tensor $\hat{R}$ on $D^s(M)$ and $\bar{R}$ on $D^s_\mu(M)$ are also defined in the right-invariant way. First, the curvature tensor $\hat{R}$ is defined on $D^s(M)$ by

$$(\hat{R}(U, V)W)_\eta := (R(U \circ \eta^{-1}, V \circ \eta^{-1})W \circ \eta^{-1}) \circ \eta, \tag{114}$$

where the R is the curvature tensor on M for $u, v, w, \in T_e D^s_\mu(M)$:

$$R(u, v)w := \nabla_u(\nabla_v w) - \nabla_v(\nabla_u w) - \nabla_{[u,v]} w \tag{115}$$

(see (60)). The curvature tensor $\bar{R}$ on $D^s_\mu(M)$ is given by the right-hand side of (114) but ∇ replaced with $\bar{\nabla}$ in (115). Both of the curvature tensors $\hat{R}$ and $\bar{R}$ are related by the following Gauss–Codazzi equation (70):

$$\begin{aligned}\langle \hat{R}(U, V)W, Z\rangle_{L^2} = \langle \bar{R}(U, V)W, Z\rangle &+ \langle S(U, W), S(V, Z)\rangle \\ &- \langle S(U, Z), S(V, W)\rangle.\end{aligned} \tag{116}$$

7.3. Formulation of hydrodynamics

7.3.1. Hydrodynamic connection

Let us consider the geodesic equation on the manifold $D^s_\mu(M)$, the group of volume-preserving diffeomorphisms of M. This is a mathematical derivation of the hydrodynamic equation of an inviscid incompressible fluid. In other words, this is an application of the geometrical theory in the theoretical physics to the dynamical systems. Because of the right-invariance of the metric on $TD^s_\mu(M)$ defined in the previous section, the Riemannian connection satisfies the expression like (58). In the present case, using the symbol $\bar{\nabla}$ for the induced connection, we obtain

$$2\langle \bar{\nabla}_u v, w\rangle = \langle [u, v], w\rangle - \langle [v, w], u\rangle + \langle [w, u], v\rangle \tag{117}$$

for $u, v, w \in T_e D^s_\mu(M)$. The commutator of the present problem is given by

$$[u, v](s) = \bar{\nabla}_u v - \bar{\nabla}_v u \tag{118}$$

(see (37) and (38)), where the right-hand side is divergence-free too. Introducing the adjoint operator $\mathrm{ad}_v\, w = [v, w]$ and the coadjoint operator by $\langle \mathrm{ad}_v^*\, u, w\rangle = \langle u, \mathrm{ad}_v\, w\rangle = \langle u, [v, w]\rangle$, we obtain from (117)

$$\bar{\nabla}_u v = \mathrm{P}\left[\frac{1}{2}([u, v] - \mathrm{ad}_u^* v - \mathrm{ad}_v^* u) + \mathrm{grad}\, f\right],$$

by the non-degeneracy of the metric (Section 2.3). The function f is naturally introduced because of the divergence-free of the tangent vector w. In fact, $\langle \mathrm{grad}\, f, w\rangle = 0$ for any scalar function $f(x)$, which is to be determined so as to satisfy the condition, $\mathrm{div}\, \bar{\nabla}_u v = 0$.

7.3.2. Formulas in $\mathbb{R}^3$ space

In the space $\mathbb{R}^3$, for $\boldsymbol{u}, \boldsymbol{v} \in T_e D_\mu(\mathbb{R}^3)$,

$$\mathrm{ad}_{\boldsymbol{u}}\, \boldsymbol{v} := [\boldsymbol{u}, \boldsymbol{v}] = [\mathscr{L}_{\boldsymbol{u}}, \mathscr{L}_{\boldsymbol{v}}] = (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{u},$$

(see (33), (37), (38)), where $\boldsymbol{\nabla} = (\partial_1, \partial_2, \partial_3)$, $\boldsymbol{u}\cdot\boldsymbol{\nabla} = u_1\partial_1 + u_2\partial_2 + u_3\partial_3$ and $\nabla\cdot\boldsymbol{u} = 0$, $\nabla\cdot\boldsymbol{v} = 0$. The definition relation $\langle \mathrm{ad}_{\boldsymbol{u}}\cdot, \boldsymbol{v}\rangle = \langle \cdot, \mathrm{ad}_{\boldsymbol{u}}^*\, \boldsymbol{v}\rangle$ leads to, after integration by parts,

$$\mathrm{ad}_{\boldsymbol{u}}^* \boldsymbol{v} = -(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v} - v^k\boldsymbol{\nabla}u^k = \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) - \boldsymbol{\nabla}h - \boldsymbol{\nabla}f_{uv}$$

for a function f_{uv}, where $h = u^k v^k = \boldsymbol{u}\cdot\boldsymbol{v}$. Thus, we obtain

$$\begin{aligned}\bar{\nabla}_{\boldsymbol{u}}\boldsymbol{v} &= \mathrm{P}\left[\frac{1}{2}([\boldsymbol{u}, \boldsymbol{v}] - \mathrm{ad}_{\boldsymbol{u}}^*\boldsymbol{v} - \mathrm{ad}_{\boldsymbol{v}}^*\boldsymbol{u}) + \mathrm{grad}\, f\right]\\ &= \mathrm{P}\left[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \frac{1}{2}\boldsymbol{\nabla}h + \boldsymbol{\nabla}f + \boldsymbol{\nabla}f_{uv} + \boldsymbol{\nabla}f_{vu}\right]\\ &= \mathrm{P}[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{\nabla}p],\end{aligned}$$

where $p = (1/2)h + f + f_{uv} + f_{vu}$ and $\mathrm{div}\, \bar{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = 0$. In particular, setting $\boldsymbol{v} = \boldsymbol{u}$, we have

$$\bar{\nabla}_{\boldsymbol{u}}\boldsymbol{u} = \mathrm{P}[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla}p]. \tag{119}$$

7.3.3. Geodesic equation

The geodesic equation in the right-invariant time-dependent problem must be considered according to the formulation of Section 3.8.2. Consider a curve $t \to \eta_t = \eta$ and its tangent $\dot{\eta}_t$. Using (110) and (54) with $(X = Y =) U_e = V_e$ and $U_e = \dot{\eta}\circ\eta^{-1}$, the right-invariant connection of a time-dependent problem is given by

$$(\hat{\nabla}_{U_\eta} U_\eta)_\eta = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\eta}\circ\eta^{-1})\circ\eta + (\nabla_{\dot{\eta}\circ\eta^{-1}}\dot{\eta}\circ\eta^{-1})\circ\eta. \tag{120}$$

Using the definition of geodesic equation (78) and representation (107), the present form of the geodesic equation is

$$0 = \mathrm{P}[(\hat{\nabla}_{\dot{g}_t}\dot{g}_t)_{g_t}] = \mathrm{P}[\partial_t u + \nabla_u u]\circ g_t, \tag{121}$$

where $\operatorname{div} u = 0$. The Euler equation of motion of an incompressible fluid in $\mathbb{R}^3$ is obtained by right translation g_t^{-1} to $e = g_0$ and substituting (119):

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla} p = 0, \tag{122}$$

where the function p is determined so as to satisfy the condition $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$.

The Jacobi equation (84) is rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \|J\| = (\|\nabla_T \boldsymbol{e}_J\|^2 - K(T, \boldsymbol{e}_J))\|J\|, \tag{123}$$

where $T = \partial g/\partial t$ is the tangent and the Jacobi vector is $J = \partial g/\partial \alpha = \|J\|\boldsymbol{e}_J$ for a varied family of geodesic curves $g(t, \alpha)$, and $K(T, \boldsymbol{e}_J)$ is the sectional curvature:

$$K(T, \boldsymbol{e}_J) := \langle \bar{R}(T, \boldsymbol{e}_J)\boldsymbol{e}_J, T\rangle = R_{ijkl} T^i e_J^j T^k e_J^l.$$

7.3.4. *Jacobi field as a frozen field*

Let us consider the Jacobi field from a different point of view. According to the above definitions of T and J, Eq. (79) is written as $\bar{\nabla}_T J = \bar{\nabla}_J T$ (see (59)). Therefore, both of the vector fields T and J commute by the torsion-free property (55), and therefore the Lie derivative vanishes:

$$\mathscr{L}_T J = [T, J] = \bar{\nabla}_T J - \bar{\nabla}_J T = 0 \tag{124}$$

(see (42)). The argument just above Eq. (67) asserts that the torsion-free is valid not only with the connection $\bar{\nabla}$, but also with the connection $\hat{\nabla}$ as well. Thus, we have $\hat{\nabla}_T J = \hat{\nabla}_J T$.

In time-dependent problems in $\mathbb{R}^3$ space, this is rewritten as [17]

$$\partial_t \boldsymbol{J} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{J} = (\boldsymbol{J} \cdot \boldsymbol{\nabla})\boldsymbol{u}, \tag{125}$$

(equivalent to (42)). Assuming $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$, [18] this equation is transformed to

$$\partial_t \boldsymbol{J} + \boldsymbol{\nabla} \times (\boldsymbol{J} \times \boldsymbol{u}) = 0, \tag{126}$$

because $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. This is usually called the equation of *frozen* field, since it describes that the vector field $\boldsymbol{J}$ is carried along with the flow $\boldsymbol{u}$ and behaves as if $\boldsymbol{J}$ was frozen to the carrier fluid. If the flow u_t is represented by the map $\phi_t(x) = y_t(x)$, then the $\boldsymbol{J}(t) = (J^\alpha)$ is given by the Cauchy solution (see Remark of Section 2.6.2):

$$J^\alpha(t) = \frac{\partial y_t^\alpha}{\partial x^k} J^k(0).$$

It is well-known that the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ satisfies the equation of the form (126) with $\boldsymbol{J}$ replaced by $\boldsymbol{\omega}$. The magnetic induction $\boldsymbol{B}$ in the ideal magneto-hydrodynamics is also governed by the equation of frozen field.

[17] The tangent vector (velocity vector) of a flow is written as $\hat{T} := (\partial_t, u^k \partial/\partial x^k)$, whereas the Jacobi vector (non-velocity vector) is written as $\hat{J} := (0, J^k \partial/\partial x^k)$.

[18] Eq. (125) is rewritten as $\partial_t \boldsymbol{J} + \boldsymbol{\nabla} \times (\boldsymbol{J} \times \boldsymbol{u}) = -(\boldsymbol{\nabla} \cdot \boldsymbol{J})\boldsymbol{u}$. Taking div, we obtain $\partial_t(\boldsymbol{\nabla} \cdot \boldsymbol{J}) + (\boldsymbol{u} \cdot \boldsymbol{\nabla})(\boldsymbol{\nabla} \cdot \boldsymbol{J}) = 0$, since $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. Hence, if $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$ initially, we have $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$ thereafter.

7.3.5. Interpretation of the Riemannian curvature

(i) *Time evolution of Jacobi field.* A Jacobi field $J(t)$ is uniquely determined by its value $J(0)$ and the value of $\nabla_T J$ at $t=0$ on the geodesic g_t in the neighborhood of $t=0$. Provided that $J(0)=0$ and $(\nabla_T J)_{t=0} = a_0$, it can be shown that (Hicks, 1965)

$$\frac{J(t)}{a_0} = t - \frac{t^3}{6}\frac{K(T,J)}{\|J\|^2} + O(t^4), \quad (t > 0), \tag{127}$$

where $\lim_{t\to 0} K(T,J)/\|J\|^2 = K(T_0, \boldsymbol{e}_J) \equiv K_J(0)$, and $T_0 = T(0)$. Therefore, if $K_J(0) < 0$, then $\|J(t)/a_0 t\| > 1$ for sufficiently small t, and if $K_J(0) > 0$, then $\|J(t)/a_0 t\| < 1$ for t near zero. Thus, the time development of the Jacobi vector is controlled by the curvature $K(T_0, \boldsymbol{e}_J)$ and in particular by its sign. An L^2-distance between two corresponding points (having the same time t) on two geodesics $g_t(\boldsymbol{x} : v_1)$ and $g_t(\boldsymbol{x} : v_2)$ starting at the same point $g_0(\boldsymbol{x} : v_1) = g_0(\boldsymbol{x} : v_2) = \boldsymbol{x}$ with different initial velocity fields v_1 and v_2 of flows in a bounded domain D is defined by

$$d(v_1, v_2 : t) := \left(\int_D |g_t(\boldsymbol{x} : v_1) - g_t(\boldsymbol{x} : v_2)|^2 \, \mathrm{d}\boldsymbol{x} \right)^{1/2}.$$

Evidently, one has $d(v_1, v_2 : 0) = 0$. The *distance* d is the mean L^2-distance between particles starting at the same position but evolving with the different initial velocity fields. It is shown in Hattori and Kambe (1994) that

$$d(v_1, v_2 : t) = 2\varepsilon|v'| \left(t - \frac{t^3}{6}\frac{K(\bar{v}, 2v')}{(2v')^2} \right) + O(\varepsilon^2 t, \varepsilon t^5), \tag{128}$$

where $\bar{v} = (v_1 + v_2)/2$, $\varepsilon v' = (v_1 - v_2)/2$ and ε is an infinitesimal constant. According to the definition of the Jacobi filed, $J(t) = (\partial/\partial\varepsilon) g_t(\boldsymbol{x}, : \bar{v} + \varepsilon v')|_{\varepsilon=0}$, one finds that $d \sim \varepsilon J$ for infinitesimally small ε. It is found that the sectional curvature appears as a factor to the t^3 term with a negative sign and determines the departure from the linear growth of the L^2-distance. This means, if the curvature is negative, that the L^2-distance d grows faster than the linear behavior, and furthermore infers an exponential growth of the distance $d \sim \varepsilon J$ (see (123)). The variable d is also interpreted as the distance between two neighboring particles in the same flow field (Hattori and Kambe, 1994).

In the case of fluid motions in a bounded domain D without mean flow, the particle trajectory will be folded within the domain D during its evolution. Thus, there would occur stretching (at least if $K(T,J) < 0$) and folding of the Lagrangian segment connecting two neighboring particles which would lead to *mixing* of particles and *ergodicity* of the fluid motions.

(ii) *Second fundamental form.* Let us consider how the fluid motion acquires a curvature and what the curvature is. On the group $D^s_\mu(M) = \{\eta \in D^s \colon \eta^*(\mu) = \mu\}$ of all volume preserving diffeomorphisms of M, the Jacobian operator J_x applied to $\eta(x)$ at $x \in M$ takes always the value unity:

$$D^s_\mu(M) := \{\eta \in D^s \colon J_x(\eta(x)) = 1, \ \forall x \in M\}$$

where $J : D^s(M) \to H^{(s-1)}(M)$. From the implicit function theorem, $D^s_\mu(M) = \eta = \{J_x^{-1}(1)\}$ is a *closed* submanifold of $D^s(M)$. According to the formulation of Sections 3.7 and 7.2, the difference of the two connections, $\hat{\nabla}$ in $D^s(M)$ and $\bar{\nabla}$ in $D^s_\mu(M)$, is given by the second fundamental form S of (113) (see (69)). The *curvature* of the closed submanifold $D^s_\mu(M)$ is given by $\langle \bar{R}(U,V)W,Z \rangle$ in the Gauss–Codazzi equation (116). In particular, the sectional curvature of the section spanned by the tangent vectors $X, Y \in T_\eta D^s_\mu(M)$ is given by

$$\begin{aligned}\bar{K}(X,Y)^{D_\mu} &:= \langle \bar{R}(X,Y)Y, X\rangle^{D_\mu}_{L^2} \\ &= \langle R(X,Y)Y,X\rangle^M + \langle S(X,X), S(Y,Y)\rangle - \|S(X,Y)\|^2.\end{aligned}$$

Even when the manifold M is flat, i.e. the curvature $\langle R(X,Y)Y,X\rangle^M$ vanishes, the sectional curvature $\bar{K}(X,Y)^{D_\mu}$ of the closed submanifold $D^s_\mu(M)$ does not necessarily vanish due to the second and third terms (defined as $\bar{K}_S(X,Y)$) associated with $S(X,Y)$, etc. Namely the curvature of a fluid motion in this case *originates* from the $\bar{K}_S$ part,

$$\bar{K}_S(X,Y) := \langle S(X,X), S(Y,Y)\rangle_{L^2} - \|S(X,Y)\|^2$$

for a flow of an incompressible (inviscid) fluid. Thus it is found that the restriction to the volume-preserving gives rise to the above curvature. Further it is interesting to observe that the second fundamental forms are related to the pressure gradients. In fact, we have

$$S(X,Y) := \hat{\nabla}_X Y - \bar{\nabla}_X Y = \mathtt{Q}[\hat{\nabla}_X Y].$$

In Appendix A.2, it is shown that an arbitrary vector field v can be decomposed orthogonally into a divergence-free part and a gradient part. Taking as $v = \hat{\nabla}_X Y$, one obtains immediately the form, $\mathtt{Q}[\hat{\nabla}_X Y] = \mathrm{grad}(F_\mathrm{D}(v) + H_\mathrm{N}(v))$. Thus it is found that the curvature is related to the 'grad' part of the connection $\hat{\nabla}_X Y$ which is normal to $TD^s_\mu(M)$. In particular, for $\eta \in D^s_\mu(M)$ and $\dot\eta = X$, we have

$$\begin{aligned}S(X,X) &= Q_\eta(\hat{\nabla}_{\dot\eta}, \dot\eta) = Q_\eta\left(\frac{\mathrm{d}}{\mathrm{d}t}(\dot\eta \circ \eta^{-1}) \circ \eta + (\nabla^{(\mathrm{R})}_{\dot\eta\circ\eta^{-1}} \dot\eta \circ \eta^{-1}) \circ \eta\right) \\ &= -(\mathrm{grad}\; p_X) \circ \eta,\end{aligned}$$

where p_X is the pressure of the velocity field X. Hence, the first term of $\bar{K}_S$ is represented as $\langle S(X,X), S(Y,Y)\rangle = \langle \mathrm{grad}\; p_X,\ \mathrm{grad}\; p_Y\rangle$, which is the correlation of two pressure gradients. Thus the $\bar{K}_S$ part of the curvature is given by

$$\bar{K}_S(X,Y) = \langle \mathrm{grad}\; p_X, \mathrm{grad}\; p_Y\rangle - \|\mathrm{grad}(F_\mathrm{D}(v) + H_\mathrm{N}(v))\|^2.$$

7.3.6. Space-periodic flows in a cubic space (Fourier representation)

Explicit forms are given for space-periodic flows in a cube by Fourier representation, i.e. for flows on the flat 3-torus $M = T^3 = \mathbb{R}^3/(2\pi Z)^3$ (Nakamura et al., 1992; Hattori and Kambe, 1994). With $\boldsymbol{x} \in T^3$, we have $\boldsymbol{x} = \{(x^1, x^2, x^3);\ \mathrm{mod}\, 2\pi\}$. The manifold T^3 is a bounded manifold without boundary. The elements of the Lie algebra of $D_\mu(T^3)$ can be thought of as real periodic vector fields on T^3 with divergence-free property. Such periodic fields are represented by the real part of corresponding *complex* Fourier forms. The Fourier bases are denoted by $e_k = \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ where $\boldsymbol{k} = (k_i)$ is a wave number covector $(i = 1, 2, 3)$.[19] Now the representations are complexified so that all the fields become linear (or multilinear) in the complex vector space of the complexified Lie algebra.

[19] In this section, the suffix k and other roman letters in the suffices are understood to denote three-component vectors which are written with bold faces otherwise.

The bases of this vector space are given by the functions e_k ($\boldsymbol{k} \in Z^3, \boldsymbol{k} \neq 0$). The velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ is represented as

$$\boldsymbol{u}(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} \boldsymbol{u}_k(t) e_k,$$

where $\boldsymbol{u}_k(t)$ is the Fourier amplitude, also written as $u^i(\boldsymbol{k})$ ($i = 1,2,3$). The amplitude must satisfy the two properties,

$$\boldsymbol{k} \cdot \boldsymbol{u}_k = 0, \quad \boldsymbol{u}_{-k} = \boldsymbol{u}_k^*, \tag{129}$$

to describe the solenoidal condition and reality condition respectively, where the asterisk denotes the complex conjugate. It should be noted that $\boldsymbol{u}_k$ has two independent polarization components.

Let us take four tangent fields satisfying (129): $\boldsymbol{u}_k e_k$, $\boldsymbol{v}_l e_l$, $\boldsymbol{w}_m e_m$, $\boldsymbol{z}_n e_n$. Then we have the followings. The scalar product convention such as $(\boldsymbol{u} \cdot \boldsymbol{v}) = u^1 v^1 + u^2 v^2 + u^3 v^3$ is used below. The *metric* is

$$\langle \boldsymbol{u}_k e_k, \boldsymbol{v}_l e_l \rangle = (2\pi)^3 (\boldsymbol{u}_k \cdot \boldsymbol{v}_l) \delta_{0,k+l},$$

where $\delta_{0,k+l} = 1$ (if $\boldsymbol{k} + \boldsymbol{l} = 0$) and 0 (otherwise). The *covariant derivative* is

$$\bar{\nabla}_{\boldsymbol{u}_k e_k} \boldsymbol{v}_l e_l = \mathrm{i}(\boldsymbol{u}_k \cdot \boldsymbol{l}) \frac{\boldsymbol{k}+\boldsymbol{l}}{|\boldsymbol{k}+\boldsymbol{l}|} \times \left(\boldsymbol{v}_l \times \frac{\boldsymbol{k}+\boldsymbol{l}}{|\boldsymbol{k}+\boldsymbol{l}|} \right) e_{k+l}, \tag{130}$$

where the amplitude vector on the right-hand side is perpendicular to $\boldsymbol{k} + \boldsymbol{l}$. Hence $\bar{\nabla}_\cdot \cdot$ is solenoidal (i.e. divergence free). The *commutator* is

$$[\boldsymbol{u}_k e_k, \boldsymbol{v}_l e_l] = \mathrm{i}((\boldsymbol{u}_k \cdot \boldsymbol{l})\boldsymbol{v}_l - (\boldsymbol{v}_l \cdot \boldsymbol{k})\boldsymbol{u}_k) e_{k+l},$$

the right side being also perpendicular to $\boldsymbol{k} + \boldsymbol{l}$. Hence $[\cdot,\cdot]$ is solenoidal as well. The *geodesic equation* (122) reduces to

$$\frac{\partial}{\partial t} u^l(\boldsymbol{k}) + \mathrm{i} \sum_{\boldsymbol{p}+\boldsymbol{q}} \sum_{\boldsymbol{k}_{m,n}} \left(\delta_{ln} - \frac{k_l k_n}{k^2} \right) k_m u^m(\boldsymbol{p}) u^n(\boldsymbol{q}) = 0, \tag{131}$$

by using (130). From definitions (114)–(116) and (130), the *curvature tensor* is

$$\begin{aligned} \bar{R}_{klmn} &= \langle \bar{R}(\boldsymbol{u}_k e_k, \boldsymbol{v}_l e_l) \boldsymbol{w}_m e_m, \boldsymbol{z}_n e_n \rangle \\ &= (2\pi)^3 \left(\frac{(\boldsymbol{u}_k \cdot \boldsymbol{m})(\boldsymbol{w}_m \cdot \boldsymbol{k})}{|\boldsymbol{k}+\boldsymbol{m}|} \frac{(\boldsymbol{v}_l \cdot \boldsymbol{n})(\boldsymbol{z}_n \cdot \boldsymbol{l})}{|\boldsymbol{l}+\boldsymbol{n}|} - \frac{(\boldsymbol{v}_l \cdot \boldsymbol{m})(\boldsymbol{w}_m \cdot \boldsymbol{l})}{|\boldsymbol{l}+\boldsymbol{m}|} \frac{(\boldsymbol{u}_k \cdot \boldsymbol{n})(\boldsymbol{z}_n \cdot \boldsymbol{k})}{|\boldsymbol{k}+\boldsymbol{n}|} \right), \end{aligned}$$

for $\boldsymbol{k}+\boldsymbol{l}+\boldsymbol{m}+\boldsymbol{n} = 0$ only and vanishes otherwise (derived from definitions (113)–(116) and (130)). The cases when the denominator vanishes should be excluded. It is to be noted that the above formulas reduce to those of Arnold (1966) when two-dimensionality is imposed.

As an application, a flow with Beltrami property is considered, that is, we assume that the velocity field $U_{\boldsymbol{p}} = \boldsymbol{u}_{\boldsymbol{p}} e_p + \boldsymbol{u}_{-\boldsymbol{p}} e_{-p} = 2\,\mathrm{Re}[\boldsymbol{u}_{\boldsymbol{p}} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}]$ satisfies the Beltrami condition, $\nabla \times U_{\boldsymbol{p}} = \lambda U_{\boldsymbol{p}}$, $\lambda \in \mathbb{R}$. This eigenvalue problem can be solved with $\lambda^2 = |\boldsymbol{p}|^2$. It is readily shown that $U_{\boldsymbol{p}}$ is a steady-state solution. Let $X = \sum \boldsymbol{v}_l e_l$ be any velocity field satisfying (129). Then one obtains $K(U_{\boldsymbol{p}}, X) = \{\text{negative terms}\}$ (Nakamura et al., 1992). This result of negative sectional curvature is a three-dimensional counterpart of the Arnold's two-dimensional finding (1966). The negative sectional curvature leads to exponential growth of the Jacobi vector $\|J\|$ according to (123) (see also (127), (128)). This means that the distance between the two neighboring geodesics also grows exponentially.

8. Motion of a vortex filament

The dynamics of an isolated thin vortex filament, embedded in an ideal incompressible fluid, is known to be well-approximated by the *local induction approximation*, LIA (Saffman, 1992) when the filament curvature is sufficiently small. A vortex filament is assumed to be spatially periodic and given by a time-dependent C^∞-curve $\boldsymbol{x}(s,t)$ in $\mathbb{R}^3$ with $s \in S^1$ the length parameter and t the time parameter, that is, $\boldsymbol{x} : S^1 \times \mathbb{R} \to \mathbb{R}^3$. As illustrated just below, this system is characterized with the rotation group $G = \mathrm{SO}(3)$ pointwise on the S^1 manifold (Kambe, 1998; Suzuki et al., 1996). The group $G(S^1)$ of smooth mappings, $g : s(\in S^1) \mapsto g(s) \in G = \mathrm{SO}(3)$, equipped with the pointwise composition law, $g''(s) = g'(s) \circ g(s)$ for $g, g', g'' \in G$, is an infinite-dimensional Lie group, i.e. a *loop group*. The corresponding loop algebra leads to the Landau–Lifshitz equation, which is derived as the geodesic equation (Section 8.2). Further, the loop-group formulation admits a central extension (de Azcárraga and Izquierdo, 1995) in Section 8.3. This section includes a new formulation for the geodesic equations of motion of a vortex filament on the basis of the theory of loop group and its extension.[20] This gives a new interpretation to the local induction equation and the equation of Fukumoto and Miyazaki (1991) from a geometrical point of view.

8.1. Local induction equation

Suppose that motion of a vortex filament is governed by the LIA, namely,

$$\partial_t \boldsymbol{x} = \partial_s \boldsymbol{x} \times \partial_s^2 \boldsymbol{x}. \tag{132}$$

In the conventional mechanics terms, $\boldsymbol{x}(s,t) \in M$ is the position vector of a point on the filament. Mathematically, $\boldsymbol{x}(s,t)$ is an element of the C^∞-embeddings of S^1 into a three-dimensional (oriented) manifold M. This system is reconsidered on the basis of the Riemannian geometry. A well-known local orthonormal system, at each point $\boldsymbol{x}(s)$ on the filament at a time t, is denoted as (T, N, B), where $T(s)$ is the unit tangent $T(s) = \partial \boldsymbol{x}/\partial s$, $N(s)$ and $B(s)$ are the unit principal normal and binormal respectively. These unit vectors satisfy the Frenet–Serret equation:

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{133}$$

where $\kappa(s), \tau(s) \in \mathbb{R}$ are the curvature and torsion of the filament. The Hamiltonian of the system (132) is given by $H = \int \kappa^2(s)\,\mathrm{d}s$.

The motion of the curve $\boldsymbol{x}(s,t)$ is a map ϕ_t, that is, $x_0(s) \mapsto x_t(s) = \phi_t \circ x_0(s)$, where $x_t(s) = \boldsymbol{x}(s,t)$. Following the motion, the tangent vector $T_t(s)$ on the curve is left-translated, i.e. $T_t(s) = \partial x_t(s)/\partial s = \phi_t \circ \partial x_0(s)/\partial s = (\phi_t)_* T_0(s)$. Correspondingly, its derivatives are left-translated, e.g. $\partial_s T_t = (\phi_t)_* \partial_s T_0(s)$. Note that $\partial_s T = \kappa(s) N(s)$, the suffix t being omitted here and below. Differentiating Eq. (132) with respect to s, one obtains

$$\partial_t T = T \times \partial_s^2 T = -\partial_s^2 T \times T \tag{134}$$

[20] The derivation here for the periodic case would be generalized to the non-periodic case without much difficulty according to the footnote of Section 5 and the procedure of Section 7.

for the unit tangent vector field T. The equation of the form $\partial_t X = \Omega \times X$ describes "rotation" of the vector X with the angular velocity Ω. Hence, the factor $-\partial_s^2 T$ (=Ω, say) in the above equation is interpreted as the angular velocity at s pointwise, that is, the $\Omega = -T''$ is an element of the Lie algebra $\mathbf{so}(3)$, where the prime denotes the differentiation with respect to s. The commutator of the $\mathbf{so}(3)$ is given by the vector product of two elements (see Sections 2.6.1 and 4.1). In the sense of the pointwise locality, the group $G(S^1)$ is called the local group. As illustrated just below, the vector $T(= -\partial_s^{-2}\Omega)$ itself is interpreted as an element on the dual space $\mathbf{so}(3)^*$.

Associated with the Hamiltonian, it is useful to observe that $(\partial_s T, \partial_s T)_{R^3} = (\kappa N, \kappa N) = \kappa^2$, and further that $\Omega = -T'' \in \mathbf{so}(3)$ (Lie algebra). Then the metric of the system is defined as follows:

$$\langle \Omega, \Omega \rangle = \langle T'', T'' \rangle := \int_{S^1} (T'', AT'')_{R^3}\, \mathrm{d}s$$
$$= \int_{S^1} (\partial_s T, \partial_s T)_{R^3}\, \mathrm{d}s = \int \kappa^2\, \mathrm{d}s,$$

where $A \equiv -\partial_s^{-2}$, thus $AT'' = -T$, and $(A,B)_{R^3} = \delta_{kl}A^k B^l$. Integration by parts has been carried out at the second line. The left-translation of $\partial_s T$ noted above and the invariance of the Hamiltonian H induces the *left-invariant* property of the metric. Using $\partial_s T = -\partial_s^{-1}\Omega$, the metric is also written as $\langle \Omega, \Omega \rangle = \int_{S^1} (\partial_s^{-1}\Omega, \partial_s^{-1}\Omega)_{R^3}\, \mathrm{d}s = \int_{S^1} (\Omega, A\Omega)_{R^3}\, \mathrm{d}s$. The operator $A = -\partial_s^{-2}$ is often called the inertia operator (or momentum map) of the system.[21] Thus, it is found that, using the new symbol L for T,

$$L'' \in C^\infty(S^1, \mathbf{so}(3)) := \mathscr{L}\mathbf{g},$$
$$L(= -AL'') \in C^\infty(S^1, \mathbf{so}(3)^*),$$

where $\mathscr{L}\mathbf{g} = \mathbf{so}(3)[S^1]$ is the *loop algebra* of the loop group, $\mathscr{L}G := \mathrm{SO}(3)[S^1]$, and $C^\infty(S^1, \mathbf{so}(3)^*)$ is the dual algebra.

8.2. Loop algebra and Landau–Lifshitz equation

Now let us reformulate the above dynamical system in the following way. Let

$$X(s), Y(s), Z(s) \in \mathscr{L}\mathbf{g} = \mathbf{so}(3)[S^1] = C^\infty(s \in S^1, \mathbf{so}(3))$$

be the vector fields. Correspondingly, we define their respective dual fields by $AX, AY, AZ \in C^\infty(S^1, \mathbf{so}(3)^*)$, where $A = -\partial_s^{-2}$. The left-invariant *metric* is defined by

$$\langle X, Y \rangle := \int_{S^1} (\partial_s^{-1}X, \partial_s^{-1}Y)_{R^3}\, \mathrm{d}s = \int_{S^1} (X, AY)_{R^3}\, \mathrm{d}s.$$

The *commutator* is given by

$$[X, Y]^{(\mathrm{L})}(s) := X(s) \times Y(s)$$

(see (89)) at each s, pointwise. In the case of the invariant metric, the connection satisfies Eq. (58), and in terms of the operators ad and ad*, we have expression (77). By using the definitions of

[21] According to the theory of elliptic operators, $A = \partial^{-2}$ is defined uniquely for C^∞ functions.

$[X,Y]^{(L)} = X \times Y$ and of $\langle \mathrm{ad}_X^* Y, Z\rangle = \langle Y, [X,Z]^{(L)}\rangle$, the $\mathrm{ad}_X^* Y$ satisfies (see (91))

$$\langle \mathrm{ad}_X^* Y,\ Z\rangle = \langle Y,\ \mathrm{ad}_X Z\rangle = \int_{S^1} (AY, X\times Z)_{R^3}\,\mathrm{d}s$$
$$= \int_{S^1} (AA^{-1}(AY\times X), Z)_{R^3}\,\mathrm{d}s,$$

which leads to $\mathrm{ad}_X^* Y = A^{-1}(AY\times X) = -\partial_s^2[AY,X]^{(L)}$. It is found (Kambe, 1998; Suzuki et al., 1996) that the *connection* ∇ of the filament motion is represented by

$$\nabla_X Y := \frac{1}{2}([X,Y]^{(L)} + \partial_s^2[AX,Y]^{(L)} + \partial_s^2[AY,X]^{(L)}).$$

The *geodesic equation* ($\partial_t X + \nabla_X X = 0$) on the loop group $\mathscr{L}G = \mathrm{SO}(3)[S^1]$ is given by $\partial_t X + \partial_s^2(AX\times X) = 0$. Applying the operator $\partial_s^{-2} = -A$, we get a corresponding equation in the dual space,

$$\partial_t L - (L\times L'') = 0,$$

where $L = -AX$ and $X = L''$. Thus, we have recovered Eq. (134) for the vector $T = L$. This type of equation is called the *Landau–Lifshitz* equation (Arnold and Khesin, 1998). Further, integrating with respect to s, one gets back to Eq. (132).

The Jacobi equation is of the form (82), and the *sectional curvature* is

$$K(X,Y) = \langle R(X,Y)Y, X\rangle = \int_{S^1} f(s)\,\mathrm{d}s$$

for $X, Y \in C^\infty(S^1, \mathbf{so}(3))$, where the integrand function $f(s)$ is given explicitly in terms of $X(s), Y(s)$ and their derivatives, and the curvature is given in Kambe (1998) and Suzuki et al. (1996).[22] An example is the section spanned by the tangent $X = (0,0,1)$ of a straight-line vortex and an arbitrary tangent field Y, for which the curvature is found to be $K(X,Y) = \frac{1}{4}\int_{S^1}(AX\times Y')^2\,\mathrm{d}s \geqslant 0$, that is, the curvature is always positive.

8.3. Central extension of the algebra of filament motion

The central extension of the present vortex motion can be considered in an analogous way to the KdV problem (Sections 5.2 and 5.3). The result is the Kac–Moody algebra known in the gauge theory. However, its representation for the motion of vortex filament is not illustrated in any existing textbook. It is remarkable to find that the resulting geodesic equation is the one derived by Fukumoto and Miyazaki (1991) in the context of fluid dynamics. Let us introduce an extended algebra denoted as

$$\hat{X}, \hat{Y}, \hat{Z} \in \mathbf{so}(3)[S^1] \oplus \mathbb{R},$$

where $\hat{X} := (X,a)$, $\hat{Y} := (Y,b)$, $\hat{Z} := (Z,c)$, for $a,b,c \in \mathbb{R}$. The *metric* is defined newly by

$$\langle \hat{X}, \hat{Y}\rangle := \int_{S^1} (X, AY)_{R^3}\,\mathrm{d}s + ab$$

[22] The present X and Y correspond to X'' and Y'' in Kambe (1998).

where $A=-\partial_s^{-2}$. The *extended algebra* is defined by

$$[\hat{X},\hat{Y}]:=([X,Y]^{(\mathrm{L})}(s),c(X,Y)), \tag{135}$$

where

$$c(X,Y):=\int_{S^1}(X(s),Y'(s))_{R^3}\,\mathrm{d}s=-c(Y,X)$$

and the Jacobi identity is satisfied by the new commutator:

$$[[\hat{X},\hat{Y}],\hat{Z}]+[[\hat{Y},\hat{Z}],\hat{X}]+[[\hat{Z},\hat{X}],\hat{Y}]=0.$$

It is not difficult to show that the commutator (135) is equivalent to that of the Kac–Moody algebra (de Azcárraga and Izquierdo, 1995). The extended *connection* is found to be given by

$$\nabla_{\hat{X}}\hat{Y}=\left(\nabla_X Y,\frac{1}{2}\int_{S^1}(X,\partial_s Y)_{R^3}\,\mathrm{d}s\right),$$

$$\nabla_X Y\equiv\frac{1}{2}([X,Y]^{(\mathrm{L})}+\partial_s^2[AX,Y]^{(\mathrm{L})}+\partial_s^2[AY,X]^{(\mathrm{L})}-\partial_s^2(a\partial_s Y+b\partial_s X)).$$

Then the *geodesic equation* ($\partial_t\hat{X}+\nabla_{\hat{X}}\hat{X}=0$) for the extended system is obtained as

$$\partial_t X+\partial_s^2(AX\times X)-a\partial_s^3X=0,$$

$$\partial_t a=0.$$

The second equation is follows from the property, $\int_{S^1}(X,\partial_s X)_{R^3}\,\mathrm{d}s=0$. Applying the operator A, we obtain the equation for $\boldsymbol{x}_s=L=-AX$ ($X=L''$):

$$\partial_t L-(L\times L'')-a\partial_s^3L=0.$$

Integrating this with respect to s, we get back to the equation for the *space curve* $\boldsymbol{x}(s,t)$:

$$\boldsymbol{x}_t=\boldsymbol{x}_s\times\boldsymbol{x}_{ss}+a\boldsymbol{x}_{sss},$$

in $\mathbb{R}^3$. Reparameterizing s to make it to be the arc-length, one finds $\boldsymbol{x}_t=\boldsymbol{x}_s\times\boldsymbol{x}_{ss}+a(\boldsymbol{x}_{sss}+(3/2)\kappa^2\boldsymbol{x}_s)$, where $\kappa(s)=(\boldsymbol{x}_{ss},\boldsymbol{x}_{ss})^{1/2}$ is the curvature of the filament at a point s. The shape of the filament is not changed by the additional term.

This is equivalent to the equation of Fukumoto and Miyazaki (1991), which was originally derived for the motion of a thin vortex tube with an axial flow along it, This equation is called FM equation below. These are known to be the first two members of the hierarchy of completely integrable equations for the filament motion (Langer and Perline, 1991). It was shown in Section 5 that the KdV equation is a geodesic equation on the diffeomorphism group of a circle S^1 with a central extension. Here it is verified that the motion of a vortex filament governed by the LIA equation is a geodesic on the loop group $\mathscr{L}G=\mathrm{SO}(3)[S^1]$ which is SO(3)-valued with pointwise multiplication. Furthermore, the infinite-dimensional loop algebra $\mathscr{L}\mathbf{g}$ has non-trivial central extension equivalent to the Kac–Moody algebra. This is a new formulation verifying that the extended system leads to another geodesic equation with an additional third derivative term, which was derived earlier by Fukumoto and Miyazaki (1991) and shown to be a completely integrable system. It is remarkable that there is a similarity in the forms between the KdV equation and FM equation. These are two integrable systems defined over the S^1 manifold: one is a geodesic equation over the extended diffeomorphism group $\hat{D}(S^1)$ and the other is the one over the extended loop group $\hat{\mathrm{SO}}(3)[S^1]$.

9. Conclusion

A geometrical theory is developed for diverse dynamical systems of both finite and infinite degrees of freedom. Based on the mathematical framework presented in Sections 2 and 3, and according to the published works by the author and others, five dynamical systems are reformulated geometrically in the subsequent sections. Although those systems are already studied with the conventional methods in physics, the present formulation provides us a deep insight into the systems and adds new geometrical characterizations of the dynamical evolutions in terms of the geodesic equations, Jacobi fields and Riemannian curvatures. The last section for the motion of a vortex filament includes a new formulation on the basis of notion of the loop algebra, disclosing an analogy between the diffeomorphic flows on a circle and the dynamics over loop groups.

Finally, it is to be remarked that, as noted already in the beginning of the Section 6 but not included, the geometrical approach can be applied to the phase transition problem. It is found in Casetti et al. (2000) that fluctuations of the configuration-space curvature exhibit a singular behavior at the phase transition. This is an evidence that the scope of geometrical theories is fundamental and very broad.

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Appendix A.

A.1. Forms and exterior multiplications

A covector is also called 1-form, which is a linear function $\omega^1(v) : \mathbb{R}^n \to \mathbb{R}$. The set of all 1-forms on $\mathbb{R}^n$ constitutes an n-dimensional dual space. Similarly, a 2-form ω^2 is defined as a function on pairs of vectors $\omega^2(v_1, v_2) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, which is bilinear and skew-symmetric with respect to two vectors v_1 and v_2.

A k-form is defined as a function of k vectors $\omega^k(v_1,\ldots,v_k):\mathbb{R}^n\times\cdots\times\mathbb{R}^n$ (k-direct-product of $\mathbb{R}^n$) $\to\mathbb{R}$, which is k-linear and skew-symmetric in the sense, $\omega^k(v_{a_1},\ldots,v_{a_k})=(-1)^\sigma\omega^k(v_1,\ldots,v_k)$ where $\sigma=0$ (if the permutation $(a_1,\ldots,a_k)$ with respect to $(1,\ldots,k)$ is even) and $\sigma=1$ (if it is odd), where $v_a=v_a^j\partial_j$.

We now introduce an exterior multiplication of two 1-forms, which associates to every pair $(\omega_\alpha^1,\omega_\beta^1)$ on $\mathbb{R}^n$ a 2-form $\omega_\alpha^1\wedge\omega_\beta^1$ on $\mathbb{R}^n\times\mathbb{R}^n$ defined by

$$\omega_\alpha^1\wedge\omega_\beta^1(v_a,v_b)=\omega_\alpha^1(v_a)\omega_\beta^1(v_b)-\omega_\alpha^1(v_b)\omega_\beta^1(v_a), \tag{A.1.1}$$

which is obviously bilinear with respect to v_a and v_b and skew-symmetric. For example, if ω_α^1 and ω_β^1 are differential 1-forms, i.e. $\omega_\alpha^1=\mathrm{d}x^i$ and $\omega_\beta^1=\mathrm{d}x^j$, then we have

$$\mathrm{d}x^i\wedge\mathrm{d}x^j(v_1,v_2)=\mathrm{d}x^i(v_1)\,\mathrm{d}x^j(v_2)-\mathrm{d}x^i(v_2)\,\mathrm{d}x^j(v_1)=\begin{vmatrix} v_1^i & v_1^j \\ v_2^i & v_2^j\end{vmatrix}, \tag{A.1.2}$$

where the last one is the determinant. In general, a *differential* k-form on $\mathbb{R}^n$ can be written in the form,

$$\omega^k=\sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k}\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}.$$

Let f be a differentiable map of an orientation-preserving diffeomorphism, $f:M_1\to M_2$, from an interior σ of M_1 onto an interior $f(\sigma)$ of M_2. Then, for any differential k-form ω^k on M_2, the following general formula of pull-back integration holds:

$$\int_{f(\sigma)}\omega^k=\int_\sigma f^*\omega^k, \tag{A.1.3}$$

which is a generalization of the integral formula (20) for 1-form. The integral of a k-form ω^k over the image $f(\sigma)$ is equal to the integral of the pull-back $f^*\omega^k$ over the original subset σ.

A.2. Function spaces L_p, H^s and orthogonal decomposition

The totality of functions, which are differentiable up to qth order with all the derivatives being continuous over the manifold M^n, is denoted by $C^q(M)$.

A function $f(x)$ is said to belong to the function space $L_p(M)$ if the integral $\int_M|f(x)|^p\,\mathrm{d}\mu(x)$ exists ($\mathrm{d}\mu$ is a volume form). The $L_2(M)$ denotes the functions which are square-integrable over the manifold M^n.

The Sobolev space $W_p^s(M)$ denotes the totality of the functions $f(x)\in L_p(M)$ which have the property $D^sf(x)\in L_p(M)$, where $x=(x_1,\ldots,x_n)$ and D^sf denotes a generalized sth derivative (in the sense of the theory of generalized functions) including the ordinary sth derivative defined by $\partial_1^{s_1}\cdots\partial_n^{s_n}f$ with $s=s_1+\cdots+s_n$.

The space $W_2^s(M)$ is written as $H^s(M)$. If $s>n/2$, then $H^s\subset C^q(M)$ by the *Sobolev's imbedding theorem*, where $q\equiv[s-n/2]$ is the maximum integer not larger than $s-n/2$. Therefore, if $s>n/2+1$, then $q\geqslant 1$ and the function $f\in H^s$ is continuously differentiable at least once.

An arbitrary vector field v on M can be decomposed orthogonally into divergence-free and gradient parts. In fact, an H^s vector field $v \in T_eM$ is written as

$$v = P_e(v) + Q_e(v),$$

where

$$Q_e(v) = \operatorname{grad} F_{\rm D} + \operatorname{grad} H_{\rm N} = \operatorname{grad}(F_{\rm D} + H_{\rm N}),$$

$$P_e(v) = v - Q_e(v).$$

The scalar functions $F_{\rm D}$ and $H_{\rm N}$ are the solutions of the following Dirichlet problem and Neumann problem, respectively,

$$\Delta F_{\rm D}(v) = \operatorname{div} v, \quad \text{where } \operatorname{supp} F_{\rm D} \subset M,$$

$$\Delta H_{\rm N}(v) = 0 \quad \text{and} \quad \langle \nabla H_{\rm N}, \boldsymbol{n} \rangle = \langle v - \nabla F_{\rm D}, \boldsymbol{n} \rangle,$$

where $\boldsymbol{n}$ is the unit normal on the boundary ∂M. There is orthogonality, $\langle \operatorname{grad} F_{\rm D}, \operatorname{grad} H_{\rm N} \rangle = 0$. Then, it can be shown that

$$\operatorname{div} P_e(v) = 0,$$

$$\langle P_e(v), Q_e(v) \rangle = 0.$$

This decomposition is called the *Helmholtz decomposition*, or *Hodge decomposition*.

A.3. Two-cocycle, the central extension and Bott cocycle

The elements of the group $D(S^1)$ describe diffeomorphisms of a circle S^1, $g : z \in S^1 \mapsto g(z) \in S^1$. We may write $z = {\rm e}^{{\rm i}x}$ and consider the map, $x \mapsto g(x)$ such that $g(x + 2\pi) = g(x) + 2\pi$ (the ϕ in the main text is written as x here), with the composition law, $g''(x) = (g' \circ g)(x) = g'(g(x))$, where $g, g', g'' \in D(S^1)$. Writing as ${\rm e}^{{\rm i}x} =: F_e(x)$, we can consider the following transformation by the mapping $x' = g(x)$:

$$\begin{aligned} F_g(x') &:= \exp[{\rm i}\eta(g)] \exp[{\rm i}g(x)] \\ &= \exp[{\rm i}\Delta(g,x)] F_e(x) = \exp[{\rm i}(\Delta(g,x) + x)], \end{aligned} \tag{A.3.1}$$

i.e. there is a phase shift $\eta(g) : D \to \mathbb{R}$ in the transformed function F_g, where

$$\Delta(g,x) = g(x) - x + \eta(g). \tag{A.3.2}$$

We are going to show that F_g is a function on $\hat{D}(S^1)$, whereas ${\rm e}^{{\rm i}g(x)}$ is a function on $D(S^1)$. The above transformation allows to define the composition law for two successive transformations as follows. For $x'' = g'x' = g'g(x)$, we may write

$$F_{g'g}(x'') = \exp[{\rm i}\Delta(g'g, x)] F_e(x). \tag{A.3.3}$$

The composition is written as

$$F_{g'}F_g(x'') = \exp[\mathrm{i}\Delta(g',x')]F_g(x') = \exp[\mathrm{i}\Delta(g',x') + \mathrm{i}\Delta(g,x)]F_e(x). \tag{A.3.4}$$

Thus, eliminating $F_e(x)$ between (A.3.3) and (A.3.4), we obtain

$$F_{g'}F_g = \omega(g',g)F_{g'g}, \quad \omega(g',g) = \exp[\mathrm{i}\Theta(g',g)], \tag{A.3.5}$$

$$\Theta(g',g) := \Delta(g',x') + \Delta(g,x) - \Delta(g'g,x). \tag{A.3.6}$$

where $\Theta(g',g)$ is called the *local* exponent. The transformation (A.3.5) is called the *projective* representation. Requiring that the associativity (associative property) holds for F_g, we obtain the following *two-cocycle condition*:

$$\omega(g'',g')\omega(g''g',g) = \omega(g'',g'g)\omega(g',g). \tag{A.3.7}$$

In fact, we have $[F_{g''}F_{g'}]F_g = \omega(g'',g')\omega(g''g',g)F_{g''g'g}$ and $F_{g''}[F_{g'}F_g] = \omega(g'',g'g)\omega(g',g)F_{g''g'g}$. The $\omega(g',g)$ satisfying (A.3.7) is called the *two-cocycle*. In terms of the exponents $\Theta(g',g)$, the two-cocycle condition reads

$$\Theta(g'',g') + \Theta(g''g',g) = \Theta(g'',g'g) + \Theta(g',g). \tag{A.3.8}$$

Substituting (A.3.2) into (A.3.6), we find

$$\Theta(g',g) = \eta(g') + \eta(g) - \eta(g'g) : D \times D \to \mathbb{R},$$

$$\omega(g',g) = \gamma_{g'}\gamma_g\gamma_{g'g}^{-1} := \omega_{\mathrm{cob}}(g',g), \quad \text{where } \gamma_g = \exp[\mathrm{i}\eta(g)].$$

With this form, the cocycle condition (A.3.7) is identically satisfied. In general, two *two-cocycles* ω and Ω are equivalent, if there exists a factor ω_{cob} such that $\omega(g',g) = \Omega(g',g)\omega_{\mathrm{cob}}(g',g)$.[23] The $\omega_{\mathrm{cob}}(g',g)$ itself is a two-cocycle corresponding to $\Omega = 1$ and called a *trivial* two-cocycle, or two-coboundary. Let us now consider briefly the problem of projective (or *ray*) representations in order to define the central extension. The ray operators $\bar{F}$ satisfy the relation:

$$\bar{F}_{g'}\bar{F}_g = \bar{F}_{g'g}, \quad g,g' \in D, \tag{A.3.9}$$

where the bar indicates the class of equivalent operators which differ in a phase $\theta \in \mathbb{R}$, i.e. $F, F' \in \bar{F} \Leftrightarrow F' = \gamma F$, where $\gamma = \mathrm{e}^{\mathrm{i}\theta}$ (said to be an element of the group $U(1)$). If a representative class F_g is selected in the class $\bar{F}_g$, the (A.3.9) will be written as $F_{g'}F_g = \omega(g',g)F_{g'g}$ like (A.3.5). Next, inside the class $\bar{F}_g$, let us take another operators written in the form, $\mathrm{e}^{\mathrm{i}\theta}F_g$ with a new variable θ. Then

$$\mathrm{e}^{\mathrm{i}\theta'}F_{g'}\mathrm{e}^{\mathrm{i}\theta}F_g = \mathrm{e}^{\mathrm{i}(\theta'+\theta)}\mathrm{e}^{\mathrm{i}\Theta(g',g)}F_{g'g} := \mathrm{e}^{\mathrm{i}\theta''}F_{g''}.$$

Suppose that, associated with a group D, we are given a local exponent $\Theta : D \times D \to \mathbb{R}$. Then, we can define a new group $\hat{D}$ consisting of elements $\hat{g} = (g,\theta)$ together with the group operation:

$$\widehat{g'} \circ \hat{g} = (g',\theta') \circ (g,\theta) = (g' \circ g, \theta' + \theta + \Theta(g',g)). \tag{A.3.10}$$

[23] The classes of inequivalent two-cocycles define the *second cohomology group* $H^2(G,U(1))$ for a group G.

In summary, the extended group $\hat{D}$ is such that: (1) it contains $U(1)$ as an invariant subgroup and $\hat{D}/U(1)=D$. In fact, the invariant subgroup $U(1)$ is a center (i.e. the element (id,θ) commutes with any element $(g,\theta')\in\hat{D}$). Namely, $\hat{D}$ is a *central extension* of D by $U(1)$; (2) D is not a subgroup of $\hat{D}$.

It can be verified that the following Bott cocycle (Bott, 1977) satisfies the two-cocycle condition (A.3.8) for the local exponent $B(g',g)$ in place of $\Theta(g',g)$:

$$B(g',g)=\frac{1}{2}\int_{S^1}\ln\partial_x(g'\circ g)\,\mathrm{d}\ln\partial_x g. \tag{A.3.11}$$

In fact, noting that $\partial_x(g'\circ g)(x)=\partial_x(g'(g(x))=g'_x g_x$, its right-hand side is

$$\begin{aligned}B(g',g)+B(g'',g'g)&=\frac{1}{2}\int_{S^1}[\ln(g'_x g_x)\,\mathrm{d}\ln(g_x)+\ln(g''_x g'_x g_x)\,\mathrm{d}\ln(g'_x g_x)]\\&=\frac{1}{2}\int_{S^1}[\ln(g'_x)\,\mathrm{d}\ln(g_x)+\ln(g''_x)\,\mathrm{d}[\ln(g'_x)+\ln(g_x)]],\end{aligned}$$

since $\ln(g'_x g_x)=\ln(g'_x)+\ln(g_x)$ and $\int_{S^1}\ln(g_x)\,\mathrm{d}\ln(g_x)=\int_{S^1}\mathrm{d}(\ln(g_x))^2/2=0$. Similarly, the left-hand side is

$$B(g'',g')+B(g''g',g)=\frac{1}{2}\int_{S^1}[\ln(g''_x)\,\mathrm{d}\ln(g'_x)+\ln(g'_x g'_x)\,\mathrm{d}\ln(g_x)].$$

Thus, it is found that the two-cocycle condition $B(g',g)+B(g'',g'g)=B(g'',g')+B(g''g',g)$ is satisfied.

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